THE TRANSMUTED GENERALIZED LINDLEY DISTRIBUTION: PROPERTIES AND AN APPLICATION TO A DATA SET ON TIME-UP-TO-CURE OF PATIENTS TREATED WITH A TRIAZOLE ANTIFUNGAL DRUG IN AN INTENSIVE CARE UNIT

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▪ ABSTRACT: In this paper, we consider transmuted generalized Lindley distribution, obtained via the quadratic rank transmutation map under the Lindley distribution. This distribution exhibits, in addition to decreasing, increasing and bathtub hazard rates, depending on its parameters, also unimodal hazard rate shape. A comprehensive mathematical treatment of this distribution is provided. Expressions for the moment generating function, moments, order statistics, residual life and reversed failure rate function are derived. The model parameters are estimated by the maximum likelihood method. A simulation study is performed to verify the behavior of the estimation procedure in terms of mean square errors and coverage probability. Global and local influence diagnostic procedures are provided. We then analyse a real data set on time-up-to-cure of patients treated with a triazole antifungal drug in an intensive care unit in Brazil.

Keywords: Lindley distribution; maximum likelihood method; transmutation map; influence analysis.

1 Introduction

In many applied sciences such as medicine, engineering and finance, amongst others, modeling and analyzing lifetime data is crucial. Several lifetime distributions

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have been used to model such kinds of data. For instance, the exponential, Weibull, gamma, Rayleigh distributions and their generalizations (GUPTA; KUNDU, 1999). Each distribution has its own characteristics due specifically to the shape of the failure rate function, which may be only monotonically decreasing or increasing or constant in its behavior, as well as nonmonotone, being bathtub shaped or even unimodal.

Here we consider a real data on time-up-to-cure of patients treated with a triazole antifungal drug in an intensive care unit. The fluconazole is a triazole antifungal drug used in the treatment and prevention of superficial and systemic fungal infections. This antifungal is used as a empirical anticandidal therapy in patients in Intensive Care Unit (ICU) once at least 1% to 2% of all ICU patients develop invasive cadidiasis at some point during their stay. The mortality rate attributable to candidemia and invasive infection with *Candida* specie at other normally sterille sites exceed 30% to 40%, and invasive candidiasis is associated with increased length of ICU stay and health care costs (SCHUSTER et al., 2008).

Our data set consists of time-up-to-cure of 54 patients treated in 2010 with this drug in an ICU at University Hospital, in Maringá city, Paraná State, Brazil. All patients were treated for a period of 1 to 39 days.

In order to verify the possible shape for the hazard function Figures 1, left panel, shows the TTT plots. Interested readers can refer to Barlow and Campo (1975) for more information on TTT plotting. Overall, if the TTT plot is concave it indicates increasing hazard, which is our case. Furthermore, the boxplot in Figure 1 right panel, shows the distribution of the treatment times (times-up-to-cure).

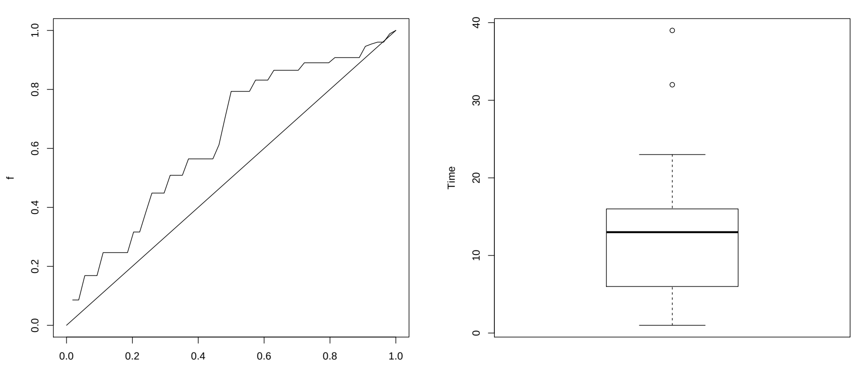

Figure 1 - Left panel: TTTPlot of times; Right panel: boxplot of treatment times.

The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distributions. Because of this, considerable effort has been expended in the development of large classes of standard probability distributions along with relevant statistical methodologies, since there still remain many important problems where the real data does not follow any of the classical or standard lifetime probability distributions.

In this paper, we consider a new lifetime distribution by transmuting and compounding the generalized Lindley distribution, hereafter Transmuted Generalized Lindley (TGL) distribution. Briefly, it is the functional composition of a cumulative distribution function on a distribution with the inverse cumulative distribution (quantile) function of a non-Gaussian distribution (SHAW; BUCKLEY, 2007). In this case, it incorporates a new third parameter (in our case λ), which introduces a skewness while preserving the moments of the distribution base (SHAW; BUCKLEY, 2007; GRANZOTTO; LOUZADA, 2015; GRANZOTTO; LOUZADA; BALAKRISHNAN, 2017).

The paper is organized as follows. A background of the Lindley and its generalization are presented in Section 2 beyond the genesis of the transmutation map. The derivation of the transmuted generalized Lindley distribution is presented in Section 3. The important properties such as moments, moment generating function, quantiles, residual life, etc, for the transmuted Lindley distribution are presented in Section 4. In Sections 5 and 6 we presented the minimum, maximum and median order statistics, and the maximum likelihood estimates and the asymptotic confidence intervals of the unknown parameters, respectively. A simulation study performed to verify the behavior of the estimation procedure in terms of mean square errors and coverage probability is presented in Section 7. In Section 8 the new distribution is illustrated in a real data set on time-up-to-cure of patients treated with a triazole antifungal drug in an intensive care unit at the University Hospital, in Maringá city, Paraná State, Brazil. Further in this section, global and local influence diagnostic procedures are provided. Final remarks are presented in Section 8.1.

2 Background

The generalization of some well-known distributions has been considered by various authors, which also studied the various of their structural properties. Only citing some: the transmuted generalized extreme value distribution was considered by Aryal and Tsokos (2009). Aryal and Tsokos (2009) considered the transmuted Weibull distribution. Elbatal and Aryal (2013) proposed transmuted log-logistic distribution. Khan and King (2013) considered the transmuted modified Weibull distribution. Elbatal and Aryal (2013) considered the transmuted additive Weibull distribution that extends the additive Weibull distribution. Elbatal; Diab and Alim (2013)considered the transmuted modified inverse Weibull distribution. Merovci and Elbatal (2014) considered the transmuted generalized Linear Exponential Distribution. Merovci and Elbatal (2014) considered transmuted Lindley geometric distribution. Tian; Tian and Zhu (2013) considered the transmuted linear exponential distribution. Granzotto and Louzada (2015) considered a new lifetime distribution by using a quadratic rank transmutation map in order to add a new parameter to the log-logistic distribution. Lucena; Silva and Cordeiro

(2015) considered the transmuted generalized gamma distribution. Nofal et al. (2016) considered the Kumaraswamy transmuted exponentiated additive Weibull Distribution. Louzada and Granzotto (2016)considered the transmuted log-logistic regression model.

2.1 Lindley distribution

The Lindley distribution, in spite of little attention in the statistical literature, is important in the context of stress-strength reliability modeling. Besides, some researchers have proposed new classes of distributions based on modifications of the Lindley distribution, including also their properties. Lindley (1958) used a mixture of exponential and length biased exponential distributions to illustrate the different between fiducial and posterior dstributions. This mixture is called the Lindley distribution and the cumulative distribution function (c.d.f.) is given by

$$F_L(x,\theta) = 1 - (1 + \frac{\theta x}{\theta + 1})e^{-\theta x}, \qquad x > 0, \theta > 0, \tag{1}$$

and the corresponding probability density function (p.d.f.) is given by

$$f_L(x,\theta) = \frac{\theta^2}{\theta + 1}(1 + x)e^{-\theta x}, \qquad x > 0, \theta > 0, \tag{2}$$

where θ is scale parameter. Ghitany; Atieh and Nadarajah (2008 argue that the Lindley distribution could be a better lifetime model than the exponential distribution through a numerical example. In addition, they show that the hazard function of the Lindley distribution does not exhibit a constant hazard rate, indicating the flexibility of the Lindley distribution over the exponential distribution.

2.2 Generalized Lindley distribution - GL

Nadarajah and co-authors (NADARAJAH; BAKOUCH; TAHMASBI, 2011) proposed a new distribution, called Generalized Lindley (GL) distribution, for modeling lifetime data. As the authors showed in their paper, the GL distribution has better hazard rate properties than the gamma, lognormal and the Weibull distributions.

Let X be a nonnegative random variable denoting the lifetime of an individual in some population. The random variable X is said to be generalized Lindley (GL) distributed with parameters θ and α if its cumulative density function (c.d.f.) is given by

$$F_{GL}(x,\theta,\alpha) = \left[1 - (1 + \frac{\theta x}{\theta + 1})e^{-\theta x}\right]^{\alpha}, \tag{3}$$

where $\theta > 0$ and $\alpha > 0$. The corresponding probability density function (p.d.f.) and the hazard (failure) rate function are given, respectively, by

$$f_{GL}(x,\theta,\alpha) = \frac{\alpha\theta^2}{\theta+1}(1+x)e^{-\theta x}\left[1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{\alpha-1}; \quad x>0, \theta>0. \quad (4)$$

and

$$h_{GL}(x,\theta,\alpha) = \frac{\alpha\theta^2(1+x)e^{-\theta x}\left[1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{\alpha-1}}{(\theta+1)\left\{1-\left[1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{\alpha}\right\}}. \quad (5)$$

Note that equation (4) has two parameters, θ and α, just like the gamma, lognormal, Weibull and exponentiated exponential distributions. Note also that equation (5) has the attractive feature of allowing for monotonically decreasing, monotonically increasing and bathtub shaped hazard rate functions though not allowing for constant hazard rate functions.

2.3 Transmutation map

In this subsection we demonstrate transmuted probability distribution. Let F_1 and F_2 be the cumulative distribution functions, of two distributions with a common sample space. The general rank transmutation as given in Shaw e Buckley (2007) is defined as

$$G_{R12}(u) = F_2(F_1^{-1}(u)) \text{ and } G_{R21}(u) = F_1(F_2^{-1}(u)).$$

Note that the inverse cumulative distribution function also known as quantile function is defined as

$$F^{-1}(y) = \inf_{x\in R}\{F(x) \geq y\} \text{ for } y \in [0,1].$$

The functions $G_{R12}(u)$ and $G_{R21}(u)$ both map the unit interval $I = [0,1]$ into itself, and under suitable assumptions are mutual inverses and they satisfy $G_{Rij}(0) = 0$ and $G_{Rij}(0) = 1$. A Quadratic Rank Transmutation Map (QRTM) is defined as

$$G_{R12}(u) = u + \lambda u(1-u), |\lambda| \leq 1, \quad (6)$$

from which it follows that the cdf's satisfy the relationship

$$F_2(x) = (1+\lambda)F_1(x) - \lambda F_1(x)^2, \quad (7)$$

which on differentiation yields,

$$f_2(x) = f_1(x)\left[(1+\lambda) - 2\lambda F_1(x)\right], \quad (8)$$

where $f_1(x)$ and $f_2(x)$ are the corresponding pdfs associated with cdf $F_1(x)$ and $F_2(x)$ respectively. An extensive information about the quadratic rank transmutation map is given in Shaw and Buckley (2007). Observe that at $\lambda = 0$ we have the distribution of the base random variable. The following Lemma proved that the function $f_2(x)$ in given (8) is a probability density function.

Lemma 2.1. *$f_2(x)$ given in (8) is a well defined probability density function.*
Proof. *Rewriting $f_2(x)$ as $f_2(x) = f_1(x)\left[(1 - \lambda(2F_1(x) - 1\right]$ we observe that $f_2(x)$ is nonnegative. We need to show that the integration over the support of the random variable is equal one. Consider the case when the support of $f_1(x)$ is $(-\infty, \infty)$. In this case we have*

$$\begin{aligned}\int_{-\infty}^{\infty} f_2(x)dx &= \int_{-\infty}^{\infty} f_1(x)\left[(1+\lambda) - 2\lambda F_1(x)\right]dx \\ &= (1+\lambda)\int_{-\infty}^{\infty} f_1(x)dx - \lambda\int_{-\infty}^{\infty} 2f_1(x)F_1(x)dx \\ &= (1+\lambda) - \lambda \\ &= 1\end{aligned}$$

Similarly, other cases where the support of the random variable is a part of real line follows. Hence $f_2(x)$ is a well defined probability density function. We call $f_2(x)$ the transmuted probability density of a random variable with base density $f_1(x)$. Also note that when $\lambda = 0$ then $f_2(x) = f_1(x)$. This proves the required result. □

3 The transmuted generalized Lindley distribution - TGL

In this Section we present the derivation of the transmuted generalized Lindley distribution, presenting its cumulative density function, probability density function, reliability function and cumulative hazard function.

Proposition 3.1. *Let X be a nonnegative random variable denoting the lifetime of an individual in some population. The random variable X is said to be transmuted generalized Lindley (TGL) with parameters θ, α and λ if its cumulative density function (c.d.f.) is given by*

$$\begin{aligned}F_{TGL}(x,\theta,\alpha,\lambda) &= G(x)\left[(1+\lambda) - \lambda G(x)\right] \\ &= \left[1 - (1 + \frac{\theta x}{\theta+1})e^{-\theta x}\right]^{\alpha} \\ &\quad \times\left\{(1+\lambda) - \lambda\left[1 - (1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{\alpha}\right\}, \qquad (9)\end{aligned}$$

and the corresponding probability density function (p.d.f.) is given by

$$\begin{aligned}f_{TGL}(x,\theta,\alpha,\lambda) &= \frac{\alpha\theta}{\theta+1}(1+x)e^{-\theta x}\left[1 - \left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}\right]^{\alpha-1} \\ &\quad \times\left[1 + \lambda - 2\lambda\left[1 - \left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}\right]^{\alpha}\right], \qquad (10)\end{aligned}$$

where $\theta, \alpha > 0$ and $\lambda \in (-1, 1)$.

Proof. *The proof is given directly by using the method presented in Section 2.3.* □

It is observed that the transmuted generalized Lindley distribution is an extended model to analyse data from complex situations and it generalizes some of the widely used distributions in reliability analysis. For instance when $\alpha = 1$ it reduces to transmuted Lindley as discussed in Merovci and Elbatal (2014). The generalized Lindley distribution is clearly a special case for $\lambda = 0$ (NADARAJAH; BAKOUCH; TAHMASBI, 2011). When $\lambda = 0$ and $\alpha = 1$ then the resulting distribution is an Lindley distribution, see for example (GHITANY; ATIEH; NADARAJAH, 2008)).

The reliability function of the TGL model is denoted by $R_{TGL}(t)$ and is defined as

$$\begin{aligned} R_{TGL}(t) &= 1 - F_{TGL}(t) \\ &= 1 - \left[1-(1+\frac{\theta t}{\theta+1})e^{-\theta t}\right]^{\alpha}\left\{(1+\lambda)-\lambda\left[1-(1+\frac{\theta t}{\theta+1})e^{-\theta t}\right]^{\alpha}\right\} \end{aligned} \quad (11)$$

For different parameters values the estimated curves can bee seen in Figures 2, upper panels. One of the characteristic in reliability analysis is the hazard rate function defined by

$$h_{TGL}(t) = \frac{f_{TGL}(t)}{1 - F_{TGL}(t)}. \quad (12)$$

It is important to note that the units for $h_{TGL}(t)$ is the probability of failure per unit of time, distance or cycles. These failure rates are defined with different choices of parameters, see Figures 2, lower panels.

The cumulative hazard function of the model is defined as

$$H_{TLG}(t) = -\ln\left|\left[1-(1+\frac{\theta t}{\theta+1})e^{-\theta t}\right]^{\alpha}\left\{(1+\lambda)-\lambda\left[1-(1+\frac{\theta t}{\theta+1})e^{-\theta t}\right]^{\alpha}\right\}\right|. \quad (13)$$

It is important to note that the units for $H_{TLG}(t)$ is the cumulative probability of failure per unit of time, distance or cycles. For all choice of parameters the distribution has the decreasing patterns of cumulative instantaneous failure rates.

4 Statistical properties

This section is devoted to the study of the statistical properties of the proposed TGL distribution.

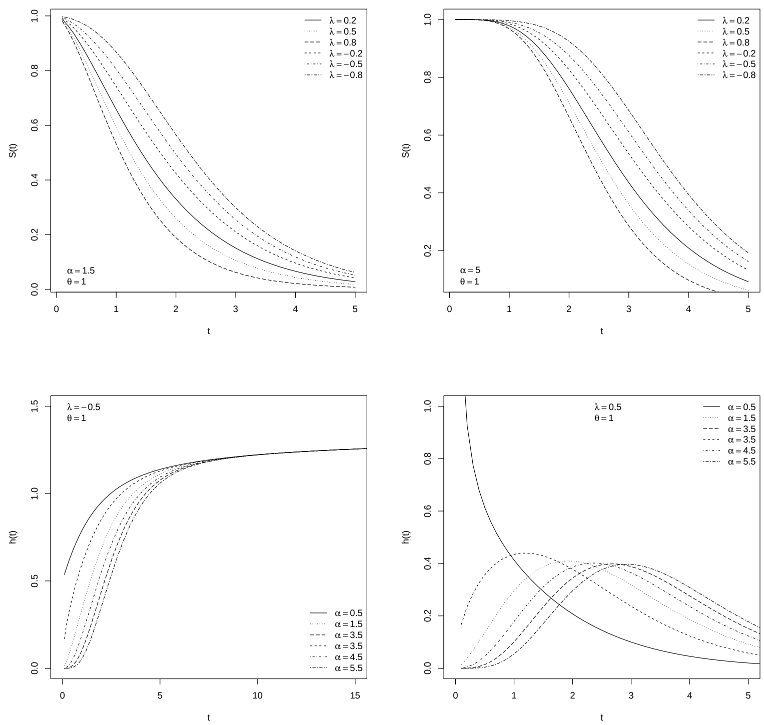

Figure 2 - Upper panels: Survival curves; Lower panels: Hazard curves.

4.1 Quantiles and random number generation

The quantile x_q of the transmuted generalized Lindley TGL$(\alpha, \theta, \lambda, x)$ is obtained from the following equation

$$F(x_q) = \left[1 - (1 + \frac{\theta x_q}{\theta + 1})e^{-\theta x_q}\right]^{\alpha} \left\{(1 + \lambda) - \lambda \left[1 - (1 + \frac{\theta x_q}{\theta + 1})e^{-\theta x_q}\right]^{\alpha}\right\} = q$$

setting $\phi = \left[1 - (1 + \frac{\theta x_q}{\theta+1})e^{-\theta x_q}\right]^{\alpha}$ then we have

$$\phi \left[(1 + \lambda) - \lambda\phi\right] = q.$$

By solving the above equation with respect to ϕ we get

$$\phi = \frac{(1 + \lambda) + \sqrt{(1 + \lambda)^2 - 4\lambda q}}{2\lambda}.$$

Hence we can obtain the quantile x_q of the transmuted generalized Lindley as follows

$$(1+\frac{\theta x_q}{\theta+1})e^{-\theta x_q} = 1-\left\{\frac{(1+\lambda)+\sqrt{(1+\lambda)^2-4\lambda q}}{2\lambda}\right\}^{\frac{1}{\alpha}}. \tag{14}$$

The above equation has no closed form solution in x_q, so we have to use a numerical technique to get the quantiles. In particular, put $q = 0.5$ in equation (14) one gets the median of TGL$(\alpha, \theta, \lambda, x)$.

Thus, random number generation as x of the TGL$(\alpha, \theta, \lambda, x)$ is defined by the following relation

$$\left[1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{\alpha}\left\{(1+\lambda)-\lambda\left[1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{\alpha}\right\} = u,$$

where $u \sim U(0,1)$. This yields,

$$(1+\frac{\theta x}{\theta+1})e^{-\theta x} = 1-\left\{\frac{(1+\lambda)+\sqrt{(1+\lambda)^2-4\lambda u}}{2\lambda}\right\}^{\frac{1}{\alpha}}. \tag{15}$$

Equation (8) above does not have a closed form solution so we generate u as uniform random variables from U$(0,1)$ and solving it for x in order to generate random numbers from the TGL distribution.

4.2 Moments

In this subsection we discuss the r_{th} moment for TGL distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

Theorem 4.1. *If X has TGL$(\alpha, \theta, \lambda, x)$ then the r_{th} moment of X is given by the following*

$$\mu_r^{'}(x) = \left\{\frac{\alpha\theta^2}{\theta+1}\sum_{j=0}^{\infty}\sum_{i=0}^{j}(-1)^j\binom{\alpha-1}{j}\binom{j}{i}\left(\frac{\theta}{\theta+1}\right)^i \right. \tag{16}$$
$$\left.\left[(1+\lambda)\binom{\alpha-1}{j}-2\lambda\binom{2\alpha-1}{j}\right]\left[\frac{\Gamma(r+i+1)}{(\theta(j+1))^{r+i+1}}\left(1+\frac{r+i+1}{\theta(j+1)}\right)\right]\right\}.$$

Proof. *Let X be a random variable with density function (10). The r_{th} ordinary* *moment of the TGL distribution is given by*

$$\begin{aligned}\mu_r^{'}(x) &= E(X^{r)} = \int_0^\infty x^r f(x)dx \\ &= \frac{(1+\lambda)\alpha\theta^2}{\theta+1}\int_0^\infty x^r(1+x)e^{-\theta x}\left[1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{\alpha-1}dx \\ &-\frac{2\lambda\alpha\theta^2}{\theta+1}\int_0^\infty x^r(1+x)e^{-\theta x}\left[1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{2\alpha-1}dx. \quad (17)\end{aligned}$$

using the series expansion

$$(1-z)^k = \sum_{j=0}^{\infty}(-1)^j\binom{k}{j}z^j, \quad (18)$$

where$|z| < 1$ *and* $k > 0$*, equation (17) can be demonstrated by*

$$\begin{aligned}\mu_r^{'}(x) &= \frac{\theta^2(1+\lambda)}{\theta+1}\sum_{j=0}^{\infty}(-1)^j\binom{\alpha-1}{j}\int_0^\infty x^r(1+x)(1+\frac{\theta x}{\theta+1})^j e^{-\theta(j+1)x}dx \\ &-\frac{2\lambda\alpha\theta^2}{\theta+1}\sum_{i=0}^{\infty}(-1)^i\binom{2\alpha-1}{i}\int_0^\infty x^r(1+x)(1+\frac{\theta x}{\theta+1})^j e^{-\theta(j+1)x}dx \quad (19)\end{aligned}$$

also applying the binomial expression for $(1+\frac{\theta x}{\theta+1})^j$ *where*

$$(1+\frac{\theta x}{\theta+1})^j = \sum_{i=0}^{j}\binom{j}{i}\left(\frac{\theta}{\theta+1}\right)^i x^i, \quad (20)$$

substituting from (20) into (19) we obtain

$$\begin{aligned}
\mu_r^{'}(x) &= \Bigg\{\frac{\alpha\theta^2(1+\lambda)}{\theta+1}\sum_{j=0}^{\infty}\sum_{i=0}^{j}(-1)^j\tbinom{\alpha-1}{j}\tbinom{j}{i}\left(\frac{\theta}{\theta+1}\right)^i \\
&\quad \times\int_0^\infty x^{r+i}e^{-\theta(j+1)x}dx+\int_0^\infty x^{r+i+1}e^{-\theta(j+1)x}dx\Bigg\} \\
&\quad -\Bigg\{\frac{2\lambda\alpha\theta^2}{\theta+1}\sum_{j=0}^{\infty}\sum_{i=0}^{j}(-1)^j\tbinom{2\alpha-1}{j}\tbinom{j}{i}\left(\frac{\theta}{\theta+1}\right)^i \\
&\quad \times\int_0^\infty x^{r+i}e^{-\theta(j+1)x}dx+\int_0^\infty x^{r+i+1}e^{-\theta(j+1)x}dx\Bigg\} \\
&= \Bigg\{\frac{\alpha\theta^2}{\theta+1}\sum_{j=0}^{\infty}\sum_{i=0}^{j}(-1)^j\tbinom{\alpha-1}{j}\tbinom{j}{i}\left(\frac{\theta}{\theta+1}\right)^i \\
&\quad \left[(1+\lambda)\tbinom{\alpha-1}{j}-2\lambda\tbinom{2\alpha-1}{j}\right]\left[\frac{\Gamma(r+i+1)}{(\theta(j+1))^{r+i+1}}\left(1+\frac{r+i+1}{\theta(j+1)}\right)\right]\Bigg\}.
\end{aligned}$$

Which completes the proof. □

We notice that if we put $\lambda = 0$, we get the r_{th} moment of generalized Lindley (NADARAJAH; BAKOUCH; TAHMASBI, 2011). Based on the first four moments of the TGL distribution, the measures of skewness $A(\Phi)$ and kurtosis $k(\Phi)$ of the TGL distribution can obtained as

$$A(\Phi)=\frac{\mu_3(\theta)-3\mu_1(\theta)\mu_2(\theta)+2\mu_1^3(\theta)}{\left[\mu_2(\theta)-\mu_1^2(\theta)\right]^{\frac{3}{2}}},$$

and

$$k(\Phi)=\frac{\mu_4(\theta)-4\mu_1(\theta)\mu_3(\theta)+6\mu_1^2(\theta)\mu_2(\theta)-3\mu_1^4(\theta)}{\left[\mu_2(\theta)-\mu_1^2(\theta)\right]^2}.$$

The moment generating function for the TGL model is derived as Theorem 4.2 shows.

Theorem 4.2. *If X has TGL distribution, then the moment generating function $M_X(t)$ has the following form*

$$\begin{aligned}
M_X(t) &= \Bigg\{\frac{\alpha\theta^2}{\theta+1}\sum_{j=0}^{\infty}\sum_{i=0}^{j}(-1)^j\tbinom{\alpha-1}{j}\tbinom{j}{i}\left(\frac{\theta}{\theta+1}\right)^i \qquad (21)\\
&\quad \left[(1+\lambda)\tbinom{\alpha-1}{j}-2\lambda\tbinom{2\alpha-1}{j}\right]\left[\frac{\Gamma(i+1)}{(\theta(j+1)-t)^{i+1}}\left(1+\frac{i+1}{(\theta(j+1)-t)}\right)\right]\Bigg\}.
\end{aligned}$$

Proof. *We start with the well known definition of the moment generating function given by*

$$
\begin{aligned}
M_X(t) &= E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx \\
&= \frac{(1+\lambda)\alpha\theta^2}{\theta+1}\int_0^\infty (1+x)e^{-(\theta-t)x}\left[1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{\alpha-1}dx \\
&\quad -\frac{2\lambda\alpha\theta^2}{\theta+1}\int_0^\infty (1+x)e^{-(\theta-t)x}\left[1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}\right]^{2\alpha-1}dx \qquad (22)
\end{aligned}
$$

using (18) and (20) into (22) we obtain

$$
\begin{aligned}
M_X(t) &= \left\{\frac{\alpha\theta^2}{\theta+1}\sum_{j=0}^{\infty}\sum_{i=0}^{j}(-1)^j\binom{\alpha-1}{j}\binom{j}{i}\left(\frac{\theta}{\theta+1}\right)^i\right. \qquad (23)\\
&\left.\left[(1+\lambda)\binom{\alpha-1}{j}-2\lambda\binom{2\alpha-1}{j}\right]\left[\frac{\Gamma(i+1)}{(\theta(j+1)-t)^{i+1}}\left(1+\frac{i+1}{(\theta(j+1)-t)}\right)\right]\right\}
\end{aligned}
$$

Which completes the proof. □

4.3 Residual life and reversed failure rate function

Given that a component survives up to time $t \geq 0$, the residual life is the period beyond t until the time of failure and defined by the conditional random variable $X - t|X > t$. In reliability, it is well known that the mean residual life function and ratio of two consecutive moments of residual life determine the distribution uniquely (GUPTA; GUPTA, 1983).

Proposition 4.3. *Let X be a nonnegative random variable distributed by a TGL distribution. The r^{th}-order moment of the residual life is given by*

$$
\begin{aligned}
\mu_r(t) &= \frac{\alpha\theta^2(1+\lambda)}{(\theta+1)\overline{F}(t)}\sum_{k=0}^{r}\sum_{j=0}^{\infty}\sum_{i=0}^{j}(-t)^{j+k}\binom{r}{k}\binom{j}{i}\left(\frac{\theta}{\theta+1}\right)^i \\
&\quad\times\left[(1+\lambda)\binom{\alpha-1}{j}-2\lambda\binom{2\alpha-1}{j}\right] \\
&\quad\times\left[\frac{1}{(\theta(j+1))^{r+i+-k+2}}\Gamma(r+i-k+2,(\theta(j+1)t))+\theta(j+1)\right. \\
&\quad\left.\times\,\Gamma(r+i-k+1,(\theta(j+1)t))\right].
\end{aligned}
$$

Proof. *In order to obtain the r^{th}-order moment of the residual life we use the general formula*

$$\mu_r(t) = E((X-t)^r \mid X > t) = \frac{1}{\overline{F}(t)}\int_t^\infty (x-t)^r f(x,\phi)dx, r \geq 1.$$

Then, applying the binomial expansion of $(x-t)^r$ and substituting $f(x,\phi)$ given by (10) into the above formula gives

$$\begin{aligned}\mu_r(t) &= \left\{\frac{\alpha\theta^2(1+\lambda)}{(\theta+1)\overline{F}(t)}\sum_{k=0}^{r}\sum_{j=0}^{\infty}\sum_{i=0}^{j}(-t)^{j+k}\tbinom{\alpha-1}{j}\tbinom{r}{k}\tbinom{j}{i}\left(\frac{\theta}{\theta+1}\right)^i\right.\\ &\left.\times\int_t^{\infty}x^{r+i-k}e^{-\theta(j+1)x}dx+\int_t^{\infty}x^{r+i-k+1}e^{-\theta(j+1)x}dx\right\}\\ &-\left\{\frac{2\lambda\alpha\theta^2}{(\theta+1)\overline{F}(t)}\sum_{k=0}^{r}\sum_{j=0}^{\infty}\sum_{i=0}^{j}(-t)^{j+k}\tbinom{2\alpha-1}{j}\tbinom{r}{k}\tbinom{j}{i}\left(\frac{\theta}{\theta+1}\right)^i\right.\\ &\left.\times\int_t^{\infty}x^{r+i-k}e^{-\theta(j+1)x}dx+\int_t^{\infty}x^{r+i-k+1}e^{-\theta(j+1)x}dx\right\}.\end{aligned}$$

Thus, the $\mu_r(t)$ is given by

$$\begin{aligned}\mu_r(t) &= \frac{\alpha\theta^2(1+\lambda)}{(\theta+1)\overline{F}(t)}\sum_{k=0}^{r}\sum_{j=0}^{\infty}\sum_{i=0}^{j}(-t)^{j+k}\tbinom{r}{k}\tbinom{j}{i}\left(\frac{\theta}{\theta+1}\right)^i\\ &\times\left[(1+\lambda)\tbinom{\alpha-1}{j}-2\lambda\tbinom{2\alpha-1}{j}\right]\\ &\times\left[\frac{1}{(\theta(j+1))^{r+i+-k+2}}\Gamma(r+i-k+2,(\theta(j+1)t))+\theta(j+1)\right.\\ &\times\Gamma(r+i-k+1,(\theta(j+1)t))\left.\right],\end{aligned}$$

where $\Gamma(s,t)=\int_t^{\infty}x^{s-1}e^{-x}dx$ is the upper incomplete gamma function. □

Also the mean residual life of the TGL distribution is given by

$$\begin{aligned}\mu(t) &= E((X-t)\mid X>t)=\frac{1}{\overline{F}(t)}\int_t^{\infty}xf(x,\varphi)dx-t\\ &= -t+\frac{\alpha\theta^2(1+\lambda)}{(\theta+1)\overline{F}(t)}\sum_{j=0}^{\infty}\sum_{i=0}^{j}(-1)^j\tbinom{j}{i}\left(\frac{\theta}{\theta+1}\right)^i\\ &\times\left[(1+\lambda)\tbinom{\alpha-1}{j}-2\lambda\tbinom{2\alpha-1}{j}\right]\\ &\times\left[\frac{1}{(\theta(j+1))^{i+3}}\Gamma(i+3,(\theta(j+1)t))+\theta(j+1)\Gamma(i+2,(\theta(j+1)t))\right].\end{aligned}$$

On the other hand, we analogously discuss the reversed residual life and some of its properties. The reversed residual life can be defined as the conditional random variable $t-X|X \leq t$ which denotes the time elapsed from the failure of a component given that its life is less than or equal to t. This random variable may also be called the inactivity time (or time since failure). Also, in reliability, the mean reversed residual life and ratio of two consecutive moments of reversed residual life characterize the distribution uniquely.

Proposition 4.4. *The r th-order moment of the reversed residual life is given by*

$$\begin{aligned} m_r(t) &= \frac{\alpha\theta^2(1+\lambda)}{(\theta+1)\overline{F}(t)}\sum_{k=0}^{r}\sum_{j=0}^{\infty}\sum_{i=0}^{j}(-t)^{j+k}\tbinom{r}{k}\tbinom{j}{i}\left(\frac{\theta}{\theta+1}\right)^i \\ &\quad\times\left[(1+\lambda)\tbinom{\alpha-1}{j}-2\lambda\tbinom{2\alpha-1}{j}\right] \\ &\quad\times\left[\frac{1}{(\theta(j+1))^{r+i+-k+2}}\gamma(r+i-k+2,(\theta(j+1)t))\right. \\ &\quad\left.+\theta(j+1)\gamma(r+i-k+1,(\theta(j+1)t))\right], \end{aligned}$$

where $\gamma(s,t)=\int_0^t x^{s-1}e^{-x}dx$ is the lower incomplete gamma function.

Proof. *The expression can be obtained directly by using the well known formula*

$$m_r(t) = E((t-X)^r \mid X \leq t) = \frac{1}{\overline{F}(t)}\int_0^t (t-x)^r f(x,\varphi)dx, r \geq 1,$$

and applying the binomial expansion of $(t-x)^r$. □

Thus the mean of the reversed residual life of the TGL distribution is given by

$$\begin{aligned} m_1(t) &= m(t) = t-\left\{\frac{\alpha\theta^2(1+\lambda)}{(\theta+1)\overline{F}(t)}\sum_{j=0}^{\infty}\sum_{i=0}^{j}(-1)^j\tbinom{j}{i}\left(\frac{\theta}{\theta+1}\right)^i\right. \\ &\quad\times\left[\frac{1}{(\theta(j+1))^{i+3}}\Gamma(i+3,(\theta(j+1)t))+\theta(j+1)\Gamma(i+2,(\theta(j+1)t))\right] \\ &\quad\left.\times\left[(1+\lambda)\tbinom{\alpha-1}{j}-2\lambda\tbinom{2\alpha-1}{j}\right]\right\}. \end{aligned}$$

Using $m(t)$and $m_2(t)$ we obtain the variance of the reversed residual life of the TGL distribution , and hence the coefficient of variation of the reversed residual life of the TGL distribution can be easily obtained.

5 Distribution of the order statistics

In fact, the order statistics have many applications in reliability and life testing. The order statistics arise in the study of reliability of a system. Let X_1, $X_2, \ldots,$

X_n be a simple random sample from TGL$(\alpha, \theta, \lambda, x)$ with cumulative distribution function and probability density function as in (9) and (10), respectively. Let $X_{(1:n)}$ $\leq X_{(2:n)} \leq \ldots \leq X_{(n:n)}$ denote the order statistics obtained from this sample. In reliability literature, $X_{(i:n)}$ denote the lifetime of an $(n-i+1)-$ out$-$ of$-$ n system which consists of n independent and identically components. Then the pdf of $X_{(i:n)}$ $, 1 \leq i \leq n$ is given by

$$f_{i::n}(x) = \frac{1}{\beta(i, n-i+1)} \left[F(x_{(i)})\right]^{i-1} \left[1 - F(x_{(i)})\right]^{n-i} f(x_{(i)}). \tag{24}$$

Moreover, the joint pdf of $X_{(i:n)}$, $X_{(j:n)}$ and $1 \leq i \leq j \leq n$ is

$$f_{i::j:n}(x_i, x_j) = C \ \left[F(x_i)\right]^{i-1} \left[F(x_j) - F(x_i)\right]^{j-i-1} \left[1 - F(x_j)\right]^{n-j} f(x_i) f(x_j), \tag{25}$$

where

$$C = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}.$$

Proposition 5.1. *Let $X_{(1:n)} \leq X_{(2:n)} \leq \ldots \leq X_{(n:n)}$ be independently identically distributed order random variables from the transmuted generalized Lindley distribution. The first order statistics $X_{(1)} = Min(X_1, X_2, ..., X_n)$ is given by*

$$\begin{aligned} f_{1:n}(x) &= n\left\{1 - \zeta_{(1)}^{\alpha}\left[(1+\lambda) - \lambda\zeta_{(1)}^{\alpha}\right]\right\}^{n-1} \\ &\quad \times \frac{\alpha\theta^2}{\theta+1}(1 + x_{(1)})e^{-\theta x_{(1)}} \ \zeta_{(1)}^{\alpha-1}\left\{(1+\lambda) - 2\lambda\zeta_{(1)}^{\alpha}\right\}, \end{aligned} \tag{26}$$

where

$$\zeta_{(i)} = 1 - (1 + \frac{\theta x_{(i)}}{\theta+1})e^{-\theta x_{(i)}} \tag{27}$$

Proof. *We start with the well known definition of the first order statistic*

$$f_{1:n}(x) = n\left[1 - F(x_{(1)})\right]^{n-1} f(x_{(1)}).$$

By using the results of the Proposition 3.1, the proof is given directly. □

Proposition 5.2. *Let $X_{(1:n)} \leq X_{(2:n)} \leq \ldots \leq X_{(n:n)}$ be independently identically distributed order random variables from the transmuted generalized Lindley distribution. The last order statistics $X_{(n)} = Max(X_1, X_2, \ldots, X_n)$ is given by*

$$\begin{aligned} f_{n:n}(x) &= n\left\{\zeta_{(n)}^{\alpha}\left[(1+\lambda) - \lambda\zeta_{(n)}^{\alpha}\right]\right\}^{n-1} \\ &\quad \times \frac{\alpha\theta^2}{\theta+1}(1 + x_{(n)})e^{-\theta x_{(n)}} \ \zeta_{(n)}^{\alpha-1}\left\{(1+\lambda) - 2\lambda\zeta_{(n)}^{\alpha}\right\} \end{aligned} \tag{28}$$

where $\zeta_{(i)}$ is given by equation (27).

Proof. *We start with the well known definition of the last order statistic*

$$f_{n:n}(x) = n\left[F(x_{(n)},\Phi)\right]^{n-1} f(x_{(n)}),\Phi).$$

By using the results of the Proposition 3.1, the proof is given directly. □

Proposition 5.3. *Let $X_{(1:n)} \leq X_{(2:n)} \leq \ldots \leq X_{(n:n)}$ be independently identically distributed order random variables from the transmuted generalized Lindley distribution. The median order statistics X_{m+1} is given by*

$$\begin{aligned} f_{m+1:n}(\widetilde{x}) &= \frac{(2m+1)!}{m!m!}\left\{\zeta^{\alpha}_{(m+1)}\left[(1+\lambda)-\lambda\zeta^{\alpha}_{(m+1)}\right]\right\}^{m} \\ &\times\left\{1-\zeta^{\alpha}_{(m+1)}\left[(1+\lambda)-\lambda\zeta^{\alpha}_{(m+1)}\right]\right\}^{m} \\ &\times\frac{\alpha\theta^2}{\theta+1}(1+x_{(m+1)})e^{-\theta x_{(m+1)}}\ \zeta^{\alpha-1}_{(m+1)}\left\{(1+\lambda)-2\lambda\zeta^{\alpha}_{(m+1)}\right\}. \end{aligned} \tag{29}$$

where $\zeta_{(i)}$ is given by equation (27).

Proof. *We start with the well known definition of the median order statistic*

$$f_{m+1:n}(\widetilde{x}) = \frac{(2m+1)!}{m!m!}(F(\widetilde{x}))^m(1-F(\widetilde{x}))^m f(\widetilde{x}).$$

By using the results of the Proposition 3.1, the proof is given directly. □

We notice that the minimum, maximum and median order statistics of three parameters transmuted generalized Lindley distribution have different life time distributions when its parameters are changed.

Now, by using the Propositions 5.1 and 5.2, the joint distribution of the the i_{th} and j_{th} order statistics can be obtained from transmuted generalized Lindley distribution and it is given by

$$\begin{aligned} f_{i::j:n}(x_i,x_j) &= C\ \left[F(x_i)\right]^{i-1}\left[F(x_j)-F(x_i)\right]^{j-i-1}\left[1-F(x_j)\right]^{n-j} f(x_i)f(x_j) \\ &= C\left\{h^{\alpha}_{(i)}\left[(1+\lambda)-\lambda h^{\alpha}_{(i)}\right]\right\}^{i-1} \\ &\times\left\{h^{\alpha}_{(j)}\left[(1+\lambda)-\lambda h^{\alpha}_{(j)}\right]-h^{\alpha}_{(i)}\left[(1+\lambda)-\lambda h^{\alpha}_{(i)}\right]\right\}^{j-i-1} \\ &\times\left\{1-h^{\alpha}_{(j)}\left[(1+\lambda)-\lambda h^{\alpha}_{(j)}\right]\right\}^{n-j} \\ &\times\frac{\alpha\theta^2}{\theta+1}(1+x_{(i)})e^{-\theta x_{(i)}}\ h^{\alpha-1}_{(i)}\left\{(1+\lambda)-2\lambda h^{\alpha}_{(i)}\right\} \\ &\times\frac{\alpha\theta^2}{\theta+1}(1+x_{(j)})e^{-\theta x_{(j)}}\ h^{\alpha-1}_{(j)}\left\{(1+\lambda)-2\lambda h^{\alpha}_{(j)}\right\}. \end{aligned} \tag{30}$$

As a special case, if $i = 1$ and $j = n$ we get the joint distribution of the minimum and maximum of order statistics

$$\begin{aligned} f_{1::n:n}(x_i, x_j) &= n(n-1)\ \left[F(x_{(n)}) - F(x_{(1)})\right]^{n-2} f(x_{(1)}) f(x_{(n)}) \\ &= n(n-1)\left\{h_{(n)}^{\alpha}\left[(1+\lambda) - \lambda h_{(n)}^{\alpha}\right]\right. \\ &\quad \left. h_{(1)}^{\alpha}\left[(1+\lambda) - \lambda h_{(1)}^{\alpha}\right]\right\}^{n-2} \\ &\quad \times \frac{\alpha\theta^2}{\theta+1}(1+x_{(1)})e^{-\theta x_{(1)}}\ h_{(1)}^{\alpha-1}\left\{(1+\lambda) - 2\lambda h_{(1)}^{\alpha}\right\} \\ &\quad \times \frac{\alpha\theta^2}{\theta+1}(1+x_{(n)})e^{-\theta x_{(n)}}\ h_{(n)}^{\alpha-1}\left\{(1+\lambda) - 2\lambda h_{(n)}^{\alpha}\right\}. \qquad (31) \end{aligned}$$

Also we can find the joint of minimum and maximum order statistics of three parameters transmuted generalized Lindley distribution when its parameters are changed.

6 Inference

In this section we consider the maximum likelihood estimators (MLE's) of TGL distribution. Let $\phi = (\alpha, \theta, \lambda)^T$,in order to estimate the parameters α, θ,and λ of transmuted generalized Lindley distribution, let $X_1, \ldots, X_n$ be a random sample of size n from TGL$(x; \alpha, \theta, \lambda)$, from equation (10) , we obtain the likelihood function as follows

$$\begin{aligned} L(\alpha,\theta,\lambda) &= \left(\frac{\alpha\theta^2}{\theta+1}\right)^n \prod_{i=1}^{n}\left\{(1+x_i)e^{\theta x_i}\left[1-\left(1+\frac{\theta x_i}{\theta+1}\right)e^{-\theta x_i}\right]^{\alpha-1}\right. \\ &\quad \left.\times\left[(1+\lambda) - 2\lambda\left[1-\left(1+\frac{\theta x_i}{\theta+1}\right)e^{-\theta x_i}\right]^{\alpha}\right]\right\}, \qquad (32) \end{aligned}$$

then the log likelihood function can be written as

$$\begin{aligned} \ln L(\alpha,\theta,\lambda) &= n\ln\alpha + 2n\ln\theta - n\ln(1+\theta) + \sum\nolimits_{i=1}^{n}\ln(1+x_i) \\ &\quad +(\alpha-1)\sum\nolimits_{i=1}^{n}\ln\left[1-\left(1+\frac{\theta x_i}{\theta+1}\right)e^{-\theta x_i}\right] - \theta\sum\nolimits_{i=1}^{n} x_i \\ &\quad +\sum\nolimits_{i=1}^{n}\ln\left[1+\lambda-2\lambda\left[1-\left(1+\frac{\theta x_i}{\theta+1}\right)e^{-\theta x_i}\right]^{\alpha}\right]. \qquad (33) \end{aligned}$$

Differentiating $\ln L(\alpha, \theta, \lambda)$ with respect to each parameter α, θ, and λ and setting the result equals to zero, we obtain maximum likelihood estimates. The partial derivatives of $\ln L(\alpha, \theta, \lambda)$ with respect to each parameter or the score function is given by

$$U_n(\phi) = (U_\alpha, U_\theta, U_\lambda)^T,$$

where

$$U_\alpha = \frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum\nolimits_{i=1}^{n} \ln\left[1 - (1 + \frac{\theta x_i}{\theta+1})e^{-\theta x_i}\right]$$
$$-2\lambda \sum\nolimits_{i=1}^{n} \frac{\ln\left(1-(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}\right)\left(1-(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}\right)^{\alpha}}{\left[1+\lambda-2\lambda\left(1-(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}\right)^{\alpha}\right]}, \quad (34)$$

$$U_\theta = \frac{\partial \ln L}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{(1+\theta)} - \sum\nolimits_{i=1}^{n} x$$
$$-(\alpha-1)\sum\nolimits_{i=1}^{n} \frac{x_i e^{-\theta x_i}\left(\frac{1}{((1+\theta))^2} - \frac{\theta x_i}{\theta+1}\right)}{\left[1-(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}\right]}$$
$$-\sum\nolimits_{i=1}^{n} \frac{2\lambda\alpha x_i e^{-\theta x_i}\left(1-(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}\right)^{\alpha-1}\left(\frac{1}{((1+\theta))^2} - \frac{\theta x_i}{\theta+1}\right)}{\left[1+\lambda-2\lambda\left(1-(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}\right)^{\alpha}\right]} \quad (35)$$

and

$$U_\lambda = \frac{\partial \ln L}{\partial \lambda} = \sum\nolimits_{i=1}^{n} \frac{1-2\left(1-(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}\right)^{\alpha}}{\left[1+\lambda-2\lambda\left(1-(1+\frac{\theta x_i}{\theta+1})e^{-\theta x_i}\right)^{\alpha}\right]}. \quad (36)$$

The maximum likelihood estimation $\widehat{\phi} = (\widehat{\alpha}, \widehat{\theta}, \widehat{\lambda})^T$ of $\phi = (\alpha, \theta, \lambda)^T$ is obtained by solving the non linear equations $U_n(\phi) = 0$. These equations cannot be solved analytically but statistical software can be used to solve them numerically. For interval estimation and hypothesis tests on the model parameters, we require the information matrix. The 3×3 observed information matrix is given by

$$I_n(\varphi) = -\begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\theta} & I_{\alpha\lambda} \\ I_{\theta\alpha} & I_{\theta\theta} & I_{\theta\lambda} \\ I_{\lambda\alpha} & I_{\lambda\theta} & I_{\lambda\lambda} \end{bmatrix},$$

where $I_n(\phi) = \partial^2 \ln L/\partial\phi\partial\phi^T$. Applying the usual large sample approximation, MLE of ϕ, i.e $\widehat{\phi}$ can be treated as being approximately $N_3(\phi, J_n(\phi)^{-1})$, where $J_n(\phi) = E\left[I_n(\phi)\right]$. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\widehat{\phi}-\phi)$ is $N_3(0, J(\varphi)^{-1})$, where $J(\phi) = \lim_{n\to\infty} n^{-1} I_n(\phi)$ is the unit information matrix. This asymptotic behavior remains valid if $J(\phi)$ is replaced by the average sample information matrix evaluated at $\widehat{\phi}$, say $n^{-1}I_n(\widehat{\phi})$. The estimated asymptotic multivariate normal $N_3(\phi, I_n(\widehat{\phi})^{-1})$ distribution of $\widehat{\phi}$ can be used to construct approximate confidence intervals for the parameters and for the hazard rate and survival functions. An $100(1-\gamma)$ asymptotic confidence interval for each parameter ϕ_r is given by$ACI_r = (\widehat{\phi}_r - z_{\frac{\gamma}{2}}\sqrt{\widehat{I_{rr}}}, \widehat{\phi}_r + z_{\frac{\gamma}{2}}\sqrt{\widehat{I_{rr}}})$, where $\widehat{I_{rr}}$ is the (r,r) diagonal element of $I_n(\widehat{\varphi})^{-1}$ for $r = 1, 2, 3$, and $z_{\frac{\gamma}{2}}$ is the quantile $1 - \frac{\gamma}{2}$ of the standard normal distribution.

7 Simulation study

In order to study the behavior of the MLEs, this section presents the results of a Monte Carlo experiment on finite samples. For that, we consider six different set of parameters for n = 30, 50, 80, 100, 150 and 300, generated according to a TGL distribution. Note that, generated for each value of the TGL distribution, we had to solve a nonlinear equation by the Newton Raphson method. All results were obtained from $1,000$ Monte Carlo replications and fixed $\theta = 1.0$ and $\alpha = 3$.

The results are summarized in two tables (the estimation process was made by using the SAS software). Table 1 shows the generated and estimated parameter values and their respectively mean square errors (MSE) over the $1,000$ MLEs, which are observed to decay as the sample size increases. Figure 3 shows the coverage probability of a 95% two sided confidence intervals for the model parameters for parameter $\lambda = -0.5, -0.2, 0.2$ and 0.5, respectively.

8 Actual data application

In this section we analyse the real data set presented in the introduction section. Firstly, we fitted three models: Lindley, GL and TGL; the fitted estimatives are presented in Table 2. In order to compare the models, we calculated the values of AIC (Akaike criteria) and weighed these values so that the result of this indicate the chance that the model i is the best among the whole set of candidate models. For the models Lindley, GL and TGL the results are, respectively, 0.2902, 0.3381 and 0.3717, providing evidence in favor of the TGL distribution.

Figure 4, left panel, shows us the empirical survival curve obtained via Kaplan-Meier method versus the estimated TGL survival curve which we can see the closeness of the two curves. In right panel we can see the increasing curve of hazard, as the TTTPlot indicated initially.

Moreover, as a goodness-of-fit procedure, we performed a global and local influence study and a residuals analysis for the TGL model by using Martingale and deviance measures.

8.1 Global and local influence

In this section we made an analysis of global and local influence for the data set given, using the TGL model and a residual analysis.

The first tool to assess the sensitivity analysis are measures of global influence. Starting with the case-deletion, that we study the effect of withdrawal

Table 1 - Estimated parameter values and their respectively mean square errors (MSE) for different values of the λ parameter and different sample sizes

Sample Size	Generated	Estimated θ	Estimated α	Estimated λ	MSE θ	MSE α	MSE λ
30	−0.2	0.990	4.008	−0.178	0.133	0.990	0.454
	−0.5	0.984	4.037	−0.441	0.115	1.031	0.455
	0.2	1.015	4.090	0.155	0.159	1.033	0.458
	0.5	1.004	4.047	0.491	0.169	1.001	0.441
50	−0.2	0.978	3.903	−0.162	0.111	0.837	0.444
	−0.5	0.974	4.005	−0.405	0.095	0.909	0.458
	0.2	1.010	3.894	0.113	0.143	0.816	0.437
	0.5	1.009	4.010	0.464	0.155	0.835	0.418
80	−0.2	0.973	3.874	−0.145	0.102	0.773	0.435
	−0.5	0.964	3.973	−0.383	0.086	0.907	0.462
	0.2	1.003	3.847	0.150	0.133	0.681	0.434
	0.5	1.005	3.941	0.453	0.155	0.703	0.412
120	−0.2	0.970	3.874	−0.134	0.097	0.732	0.417
	−0.5	0.971	3.973	−0.408	0.077	0.888	0.430
	0.2	1.006	3.874	0.133	0.135	0.653	0.418
	0.5	1.008	3.977	0.449	0.147	0.635	0.382
300	−0.2	0.978	3.878	−0.162	0.076	0.605	0.364
	−0.5	0.980	3.995	−0.431	0.054	0.754	0.342
	0.2	0.994	3.848	0.166	0.109	0.482	0.363
	0.5	1.017	3.995	0.435	0.130	0.510	0.332
600	−0.2	0.984	3.841	−0.199	0.062	0.517	0.320
	−0.5	0.990	3.966	−0.485	0.022	0.620	0.228
	0.2	0.996	3.900	0.181	0.084	0.311	0.291
	0.5	1.015	3.990	0.444	0.104	0.357	0.274

of the ith element sampled. The first measure of global influence analysis is known as generalized Cook's distance, which is defined as the standard norm of $\boldsymbol{\theta}_i = (\alpha_i, \beta_i, \lambda_i)$ and $\hat{\boldsymbol{\theta}} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ and is given by

$$CD_i(\boldsymbol{\theta}) = \left[\ \boldsymbol{\theta}_i - \hat{\boldsymbol{\theta}}\ \right]^T \left[-\ddot{L}(\boldsymbol{\theta})\right] \left[\ \boldsymbol{\theta}_i - \hat{\boldsymbol{\theta}}\ \right], \tag{37}$$

where $\ddot{L}(\boldsymbol{\theta})$ can be approximated by the estimated covariance and variance matrix. Another way to measure the global influence is through the difference in likelihoods given by

$$LD_i(\boldsymbol{\theta}) = 2\left\{l(\hat{\boldsymbol{\theta}}) - l(\boldsymbol{\theta}_i)\right\}. \tag{38}$$

Figures 5 and 6 left panels show us, respectively, the Cook's generalized distance and likelihood distances where we could see some possible influences points: $6, 8, 16, 24, 29, 34$ and 37.

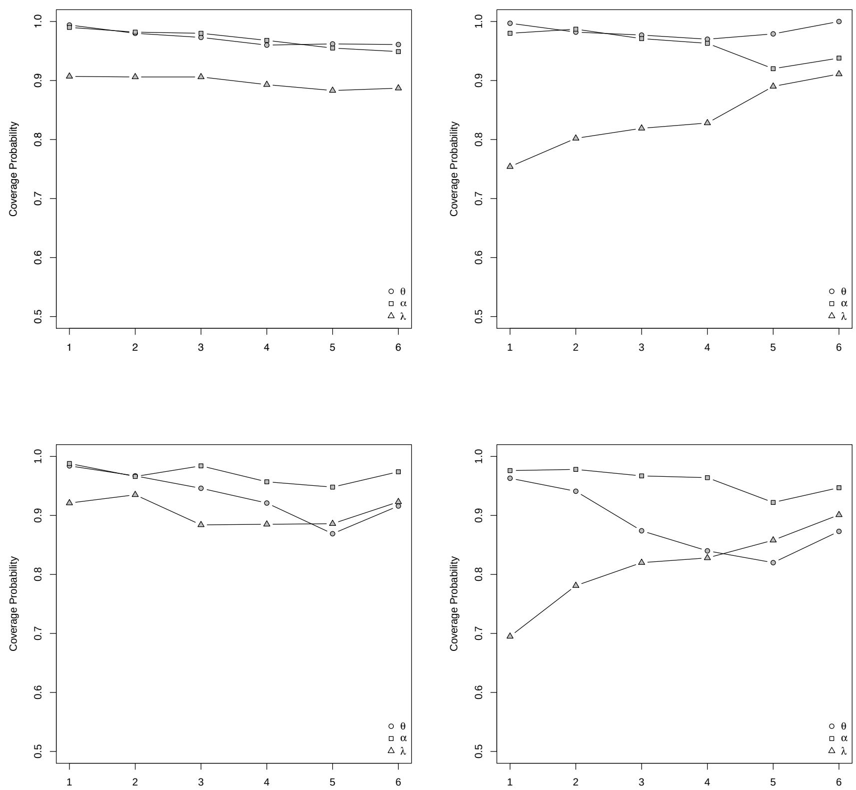

Figure 3 - Probability of coverage by considering 95% of confidence for $\theta = 1$, $\alpha = 4$ and $\lambda = -0.5$ (upper left panel), -0.2 (upper right panel), 0.2 (lower left panel) and 0.5 (lower right panel).

Furthermore, we know that the main objective of the local influence method is to evaluate changes in the results from the analysis when small perturbations are incorporated in the model and/or in the data. If such perturbations provoke disproportionate effects, it can be an indication that the model is fitted inadequately or serious departures from the assumptions of the model may exist.

In order to analyse the local influence, here we consider the response variable perturbation, ie, we will consider that each t_i is peturbed as $t_{im} = t_i + m_i S_t$, where S_t is a scale factor that may be the estimated standard deviation of T and $m_i \in \mathbb{R}$.

Table 2 - Estimatives for the parameters of the Lindley, GL and TGL models

Model	Parameter	Estimative	Standard Error	Confidence Interval 95%	
				Lower	Upper
Lindley	θ	0.1599	0.0156	0.1579	0.1618
GL	θ	0.1729	0.0235	0.1574	0.1889
	α	1.1718	0.2389	1.0232	1.3416
TGL	θ	0.1827	0.0262	0.1643	0.2002
	α	1.0773	0.3155	0.8362	1.2842
	λ	-0.2850	0.4565	-0.6280	0.0410

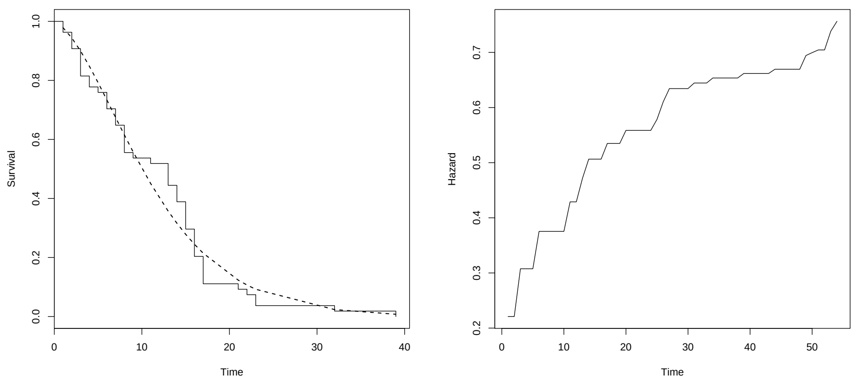

Figure 4 - Left panel: Survival curves estimated by TGL model versus empirical (by using Kaplan-Meier method); Right pane: Estimated hazard curve.

Then, the perturbed log-likelihood function becomes expressed as

$$\ln L(\alpha,\theta,\lambda \quad | \quad \boldsymbol{t},\boldsymbol{m}) = n\ln\alpha + 2n\ln\theta - n\ln(1+\theta) + \sum\nolimits_{i=1}^{n}\ln(1+t_{im})$$
$$+(\alpha-1)\sum\nolimits_{i=1}^{n}\ln\left[1-\left(1+\frac{\theta t_{im}}{\theta+1}\right)e^{-\theta t_{im}}\right] - \theta\sum\nolimits_{i=1}^{n} t_{im}$$
$$+\sum\nolimits_{i=1}^{n}\ln\left[1+\lambda-2\lambda\left[1-\left(1+\frac{\theta t_{im}}{\theta+1}\right)e^{-\theta t_{im}}\right]^{\alpha}\right].$$

After analyze the results of the perturbation, we can see that the points $16, 24, 29, 34$ and 37 are distincts of another observations (see Figures 5 and 6 right panels). Furthermore, we made a residual analyse by using the Martingale-type (39) and deviance (40) as follow:

$$r_{M_i} = 1 + \alpha\log\left(\zeta_i\right) + \log\left\{(1+\lambda) - \lambda\zeta_i^{\alpha}\right\} \tag{39}$$

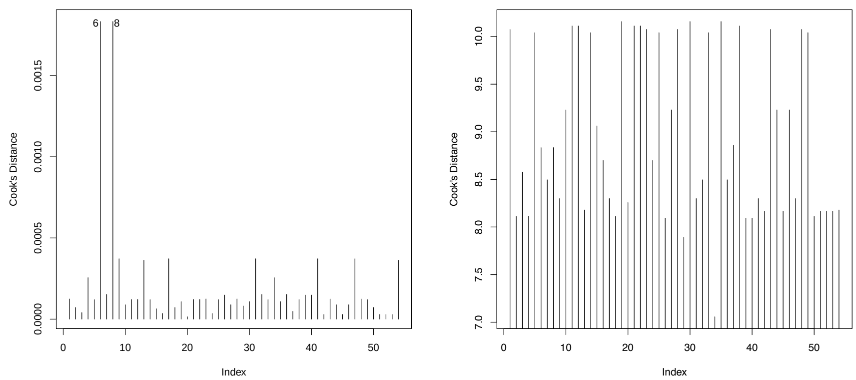

Figure 5 - Cooks's distance: influence global in left panel and in right panel after pertubation.

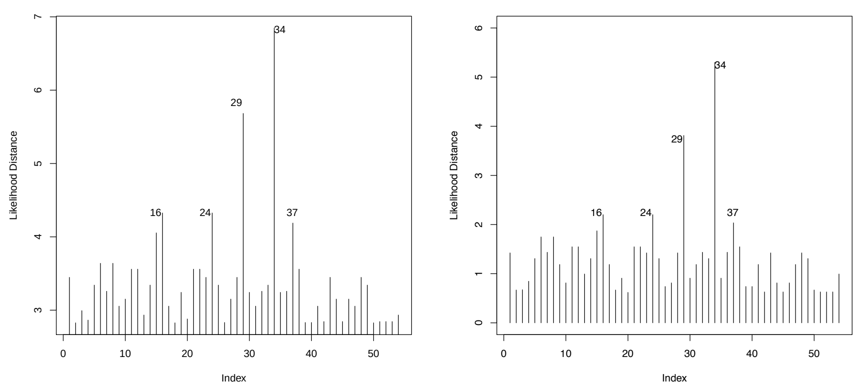

Figure 6 - Likelihood distance: influence global in left panel and in right panel after pertubation.

and

$$r_{D_i} = \text{sign}(r_{\hat{M}_i}) \left[-2(r_{\hat{M}_i} + log(1 - r_{\hat{M}_i}))\right]^{1/2}, \tag{40}$$

where $\zeta_i = 1 - \left(1 + \frac{\theta t}{\theta + 1}\right) e^{\theta t_i}$.

Figures 7 show us the results of this analyse in left and right panels, respectively.

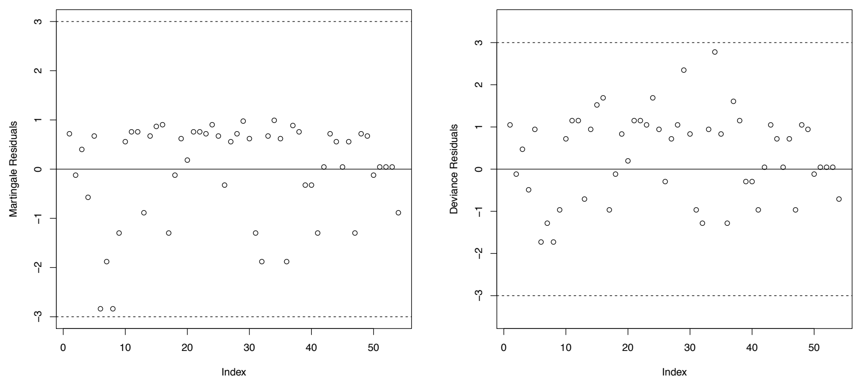

Figure 7 - Left panel: Martingale residuals; Right panel: Deviance residuals.

To reveal the impact of the detected influential observations, the RC_{θ_j} can be calculated as

$$\text{RC}_{\theta_j} = \left| \frac{\hat{\theta}_j - \hat{\theta}_{j(I)}}{\hat{\theta}_j} \right| \times 100\%, \qquad j = 1, \ldots, p+1,$$

where $\hat{\theta}_{j(I)}$ denotes the MLE of θ_j after the set I of observations has been removed. Suggested by Lee; Lu e Song (2006), the impact can bee measured by using the total and maximum relative changes and the likelihood displacement given by

$$\text{TRC} = \sum_{i=1}^{n_p} \left|\text{RC}_{\theta_j}\right|, \quad \text{MRC} = \max_j \left|\text{RC}_{\theta_j}\right| \quad \text{and} \quad LD_{(I)}(\boldsymbol{\theta}) = 2\{l(\hat{\boldsymbol{\theta}} - l(\hat{\boldsymbol{\theta}}_I)\},$$

where TRC is the total relative changes, MRC the maximum relative changes and LD the likelihood displacement, with $n_p = 3$ (the number of parameters) and $\hat{\boldsymbol{\theta}}^0$ denotes MLE of $\boldsymbol{\theta}$ after the set I of observations has been removed. Thus, we removed each of these points and fitted the model TGL again for each case as we can see in Table 3.

After the influence and residual analysis, the possible influentials observations were identified. Note that, when we removed the points $I = \{16, 24, 29, 34, 37\}$ the impact in the estimated values is high and the loss of information is low (note that, we removed 4 different times since the 16^{th} and 24^{th} are the same values). Thus, the indicated points were removed and the model TGL was fitted again as we can see in Table 4.

Is important to note that the value of likelihood estimated was 154.2010, much smaller compared to that estimated in the presence of influential points model 180.6748. Figure 8, left panel, shows us the empirical survival curve obtained via

Table 3 - RC (in %) and the corresponding TRC, MRC and $LD_{(I)}$

Case removed	Parameter	RC	TRC	MRC	$LD_{(I)}$
$\{6\} or \{8\} **$	θ	2.1984	18.2210	9.9088	7.2784
	α	9.9088			
	λ	6.1138			
$\{16\} or \{24\} **$	θ	1.9149	5.4714	2.7235	8.6506
	α	0.8329			
	λ	2.7235			
$\{29\}$	θ	5.2703	17.1133	9.1149	11.3655
	α	2.7281			
	λ	9.1149			
$\{34\}$	θ	7.9071	29.6336	17.2271	13.6961
	α	4.4993			
	λ	17.2271			
$\{37\}$	θ	1.5592	6.1659	3.9605	8.3759
	α	0.6462			
	λ	3.9605			
$\{6, 8\}$	θ	4.7477	49.0360	23.9047	15.0094
	α	23.9047			
	λ	20.3837			
$\{16, 24, 29, 34, 37\}$	θ	24.3344	63.6175	25.9083	52.9481
	α	13.3749			
	λ	25.9083			
$\{6, 8, 16, 24, 29, 34, 37\}$	θ	31.6887	88.8944	45.1855	68.1520
	α	45.1855			
	λ	12.0202			

$**$ The same value of time-up-to-cure.

Table 4 - Estimatives for the parameters of the TGL model

Parameter	Estimative	Standard Error	Confidence Interval 95%	
			Lower	Upper
θ	0.2272	0.0310	0.2063	0.2482
α	1.2214	0.3704	0.9529	1.4712
λ	-0.3588	0.3992	-0.6557	-0.0878

Kaplan-Meier method versus the reestimated TGL survival curve and; in right panel we can see the increasing curve of hazard.

After fit the final model, some statistics were obtained. The median time of hospitalization of patients who are using fluconazole at ICU was estimated at 8.8458 days. The confidence interval by considering the delta method with 95% of confidence, is given by (7.9748; 9.6033 days). The first and third estimated quantiles are given, respectively, by 5.2616 and 13.5391 days.

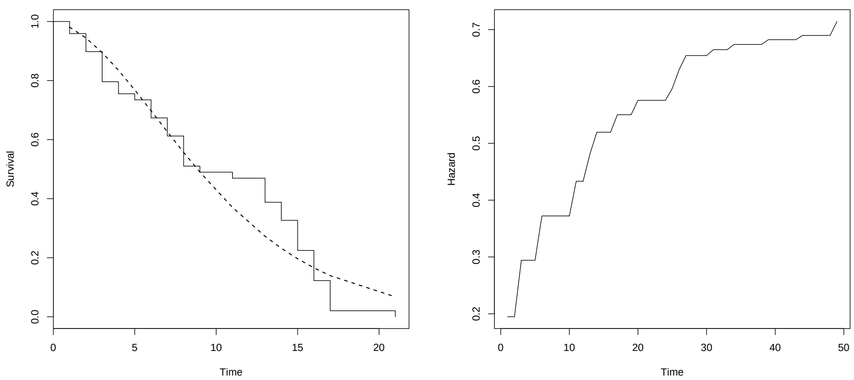

Figure 8 - Left panel: Survival curves estimated by TGL model versus empirical (by using Kaplan-Meier method); right panel: Estimated hazard curve.

Conclusions

Several lifetime distributions have been used to model such kinds of data. In this paper, in order to analyse the data set on time-up-to-cure of patients treated with a triazole antifungal drug in an intensive care unit, we developed the transmuted generalized Lindley distribution. The considered distribution was constructed by using a quadratic rank transmutation map and taking the generalized Lindley distribution with two parameters as the baseline distribution. Some mathematical properties along with order statistics and estimation issues are addressed.

A simulation study was performed to verify the behaviour of the estimation procedure in terms of mean square errors and coverage probability. Global and local influence diagnostic procedures were provided.

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LOUZADA, F.; IBRAHIM, E.; GRANZOTTO, D. C. T. Distribuição Lindley Transmutada: Propriedades e uma aplicação ao estudo do tempo até a cura de pacientes tratados com o antifúngico triazol na unidade de terapia intensiva. *Rev. Bras. Biom.,* Lavras, v.36, n.2, p.385-412, 2018.

▪ *RESUMO: Neste artigo, consideramos a distribuição de transmutada generalizada de Lindley, obtida através do mapa de transmutação de quadrática sob a distribuição de Lindley. Esta distribuição apresenta dependendo de seus parâmetros, tanto curvas de risco decrescentes e crescentes, como em forma de banheira e risco unimodal. Um tratamento matemático abrangente desta distribuição é fornecido. Expressões para a função geradora de momentos, momentos, estatísticas de ordem, vida residual e função de taxa de falha reversa são derivadas. Os parâmetros do modelo são estimados pelo método de máxima verossimilhança. Um estudo de simulação foi realizado para verificar o comportamento do procedimento de estimação em termos de erros quadráticos médios e probabilidade de cobertura do intervalo. Na aplicação, apresentamos procedimentos de diagnóstico de influência global e local. Além disso, analisamos um conjunto de dados reais sobre o tempo de cura dos pacientes tratados com um fármaco antifúngico triazol em uma unidade de terapia intensiva no Brasil.*

▪ *PALAVRAS-CHAVE: Distribuição de Lindley; método de máxima verossimilhança; mapas de transmutação; análise de influência.*

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