BAYESIAN ANALYSIS OF GENERALIZED MODIFIED WEIBULL DISTRIBUTION

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■ *ABSTRACT: Recently, Carrasco et al. (2008) proposed the modified generalized Weibull distribution (GMW). This distribution presents hazard function with bathtub, unimodal and monotonic shapes. Besides, GMW includes the Weibull, extreme value, exponentiated Weibull and generalized Rayleigh distributions as special cases. In this paper we consider a complete Bayesian analysis assuming noninformative priors for the unknown parameters and the Bayesian analysis is compared to the maximum likelihood approach. We introduce the MCMC algorithms to generate samples from the posterior distributions in order to evaluate the estimators and intervals. The numerical illustration is based on simulated and two lifetime data to illustrate the proposed methodology.*

■ *KEYWORDS: Generalized odified Weibull; Bayesian estimation; noninformative priors; MCMC methods; maximum likelihood; Jeffreys prior; covariate.*

1 Introduction

In the literature there are various probability distributions to model lifetimes of equipment or individual problems in survival analysis. Among the families of distributions used for this purpose, the most popular is the Weibull distribution which hazard function presents constant, increasing and decreasing shapes. However, when the hazard function is the unimodal or bathtub shaped, the Weibull distribution is not appropriated. Thus, in recent years, there have been proposed new distributions that fit various shapes of the hazard function and consequently to fit a larger number of practical problems. Carrasco *et al.* (2008)

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have proposed a new distribution called Generalized Modified Weibull, denoted by GMW, which hazard function can take many shapes such as constant, increasing, decreasing, unimodal and bathtub. Moreover, it has the Exponential, Weibull, Weibull Exponentiated, Extreme Value and Weibull modified distributions as special cases. They derive several properties of the GMW distribution and the maximum likelihood approach is performed to estimate the four parameters, which are reviewed in this paper.

The Generalized Modified Weibull distribution (GMW) with parameters $(\alpha, \beta, \theta, \lambda)$ has density given by,

$$f(t) = \frac{\beta t^{\theta-1}}{\alpha}(\theta + \lambda t)\exp\left\{\lambda\, t - \frac{t^{\theta}\mathrm{e}^{\lambda\, t}}{\alpha}\right\}\left[1 - \exp\left\{-\frac{t^{\theta}\mathrm{e}^{\lambda\, t}}{\alpha}\right\}\right]^{\beta-1}, \tag{1}$$

for $t > 0$; $\alpha > 0$, $\beta > 0$, $\theta \geq 0$ and $\lambda \geq 0$ where α is the scale parameter, $(\beta,\ \theta)$ denote the shape parameters and λ is an accelerating factor.

The survival and hazard functions associated to (1) are given by

$$S\,(t) = 1 - \left[1 - \exp\left\{\frac{t^{\theta}}{\alpha}\exp(\lambda t)\right\}\right]^{\beta}. \tag{2}$$

and

$$h(t) = \frac{\beta\, t^{\theta-1}\,(\theta + \lambda\, t)\exp\left\{-\frac{-\lambda\, t\alpha + t^{\theta}\mathrm{e}^{\lambda\, t}}{\alpha}\right\}\left[1 - \exp\left\{-\frac{t^{\theta}\mathrm{e}^{\lambda\, t}}{\alpha}\right\}\right]^{\beta-1}}{\alpha\left\{1 - \left[1 - \exp\left\{-\frac{t^{\theta}\mathrm{e}^{\lambda\, t}}{\alpha}\right\}\right]^{\beta}\right\}}, \tag{3}$$

respectively.

The hazard function of the GMW distribution can have various forms depending on the values of the parameters. Carrasco *et al.* (2008) show that:

if $\theta = 1$, $\beta = 1$ and $\lambda = 0$ the hazard function is constant; if $0 < \theta < 1$ and $\beta > 1$ the hazard function is monotonically decreasing; if $\theta \geq 1$ and $0 < \beta < 1$ the hazard function is monotonically increasing; if $0 < \theta < 1$ and $\beta \to \infty$ the hazard function is unimodal; if $\theta > 1$, $\lambda = 0$ and $\beta\theta < 1$ the hazard function is bathtub shaped; if $\beta = 1$, $0 < \theta < 1$ and $t^* = (-\theta + \sqrt{\theta})/\lambda$ then the hazard function is decreasing for $t < t^*$ and if $t > t^*$ the hazard function is increasing. This way, the hazard function is bathtub form.

A method for generating GMW distribution is based on inverse transform sampling. Given a random variable drawn from the uniform distribution on the interval (0, 1), then the variable

$$t^{\theta}\exp\left(\lambda t\right) + \alpha\log\left(1 - u^{1/\beta}\right) = 0 \tag{4}$$

has the GMW distribution with parameters α, β, θ and λ.

In this paper we deal with a Bayesian analysis for estimation of parameters of the Generalized Modified Weibull. Among the arguments in favour of Bayesian approach we can emphasize the possibility of incorporating prior information in the analysis. Another consideration to be highlighted is that methods based on Bayesian Inference almost always require less sample data to achieve better results. This is a very important consideration in application areas where the sample data can be costly and difficult to obtain, such as the Reliability Analysis. If the sample size is small, confidence intervals constructed using the maximum likelihood approach and normal approximation are inappropriate. When using the Bayesian approach we are required to specify a prior distribution that describes our prior knowledge about the parameters of the distribution. Prior knowledge has not been frequently considered in Bayesian analyses due to difficulty to represent the expert's opinion subjectively or arguing that, using a large amount of data, prior knowledge tends to have little effect in the final inferences. In this case, researchers usually seek to choose a prior that has little information on the parameters, allowing the data to be very informative relative to the prior information. There are in the Bayesian literature several forms of formulating noninformative priors. In most of papers in the literature it is assumed a gamma prior under the assumption of independence of the parameters. A well known noninformative prior to represent a situation with little information available a priori about the parameters was proposed by Jeffreys (1967). Since then Jeffreys prior has played an important role in Bayesian inference. Therefore, the main aim of this paper is the comparison of these priors, mainly for small sample size, for a Bayesian estimation of the Generalized Modified Weibull distribution. We show through a simulation study that Bayesian methods based on Jeffreys prior provides good estimators and confidence intervals even for small data sets.

The paper is organized as follows: in Section 2, we review the maximum likelihood estimation and in Section 3, we present a Bayesian analysis considering different noninformative priors for the parameters. Section 4 presents the results from the simulation data and in Section 5 there is a discussion about the performance of proposed approaches. In section 6, we introduce an applied example to illustrate the Bayesian approach and section 7 we consider the presence of covariates. Finally, in Section 8, we present some conclusions.

2 The Maximum Likelihood estimation

Suppose we have identically distributed lifetimes $\mathbf{t} = (t_1, ..., t_n)'$ from the GMW distribution. The likelihood function of the parameters α, β, θ and λ, based on t is given by,

$$L\left(\Theta|t\right) = \frac{\beta^n}{\alpha^n}\prod_{i=1}^{n} t_i^{\theta-1}\left(\theta+\lambda\, t_i\right)\exp\left\{\lambda\, t_i - \frac{t_i^{\theta}\mathrm{e}^{\lambda\, t_i}}{\alpha}\right\}\left[1-\exp\left\{-\frac{t_i^{\theta}\mathrm{e}^{\lambda\, t_i}}{\alpha}\right\}\right]^{\beta-1}, \tag{5}$$

and the logarithm of the likelihood function (7) is given by,

$$
\begin{aligned}
l\left(\Theta|t\right) = n\log\left(\beta\right) - n\log\left(\alpha\right) + \left(\theta-1\right)\sum_{i=1}^{n}\log\left(t_i\right) + \\
\sum_{i=1}^{n}\log\left(\theta+\lambda\,t_i\right) + \sum_{i=1}^{n}\lambda\,t_i - \sum_{i=1}^{n}\frac{t_i^{\theta}\mathrm{e}^{\lambda\,t_i}}{\alpha} - \\
\left(\beta-1\right)\sum_{i=1}^{n}\log\left[1-\exp\left\{-\frac{t_i^{\theta}\mathrm{e}^{\lambda\,t_i}}{\alpha}\right\}\right],
\end{aligned}
\tag{6}
$$

where $\Theta = (\alpha, \beta, \theta, \lambda)$.

The maximum likelihood estimators (MLE) for α, β, θ and λ are obtained by solving the following system of equations

$$
\begin{aligned}
\frac{\partial l}{\partial\alpha} &= -\frac{n}{\alpha} + \sum_{i=1}^{n}\frac{{t_i}^{\theta}\mathrm{e}^{\lambda\,t_i}}{\alpha^2} + (\beta-1)\sum_{i=1}^{n}\frac{{t_i}^{\theta}\mathrm{e}^{\lambda t_i - \frac{{t_i}^{\theta}\mathrm{e}^{\lambda\,t_i}}{\alpha}}}{\alpha^2\left(1-\mathrm{e}^{-\frac{{t_i}^{\theta}\mathrm{e}^{\lambda\,t}}{\alpha}}\right)} = 0 \\
\frac{\partial l}{\partial\beta} &= \frac{n}{\beta} + \sum_{i=1}^{n}\log\left(1-\mathrm{e}^{-\frac{{t_i}^{\theta}\mathrm{e}^{\lambda\,t_i}}{\alpha}}\right) = 0 \\
\frac{\partial l}{\partial\theta} &= \sum_{i=1}^{n}\left[\log\left(t_i\right) + \frac{1}{\theta+\lambda\,t_i}\right] - \sum_{i=1}^{n}\frac{{t_i}^{\theta}\log(t_i)\mathrm{e}^{\lambda\,t_i}}{\alpha} + (\beta-1)\sum_{i=1}^{n}\frac{{t_i}^{\theta}\log(t_i)\mathrm{e}^{\lambda t_i - \frac{{t_i}^{\theta}\mathrm{e}^{\lambda\,t_i}}{\alpha}}}{\alpha\left(1-\mathrm{e}^{-\frac{{t_i}^{\theta}\mathrm{e}^{\lambda\,t_i}}{\alpha}}\right)} = 0 \\
\frac{\partial l}{\partial\lambda} &= \sum_{i=1}^{n}\frac{t_i}{\theta+\lambda\,t_i} + \sum_{i=1}^{n}t_i - \sum_{i=1}^{n}\frac{{t_i}^{\theta+1}\mathrm{e}^{\lambda\,t_i}}{\alpha} - (\beta-1)\sum_{i=1}^{n}\frac{{t_i}^{\theta+1}\mathrm{e}^{\lambda t_i - \frac{{t_i}^{\theta}\mathrm{e}^{\lambda\,t_i}}{\alpha}}}{\alpha\left(1-\mathrm{e}^{-\frac{{t_i}^{\theta}\mathrm{e}^{\lambda\,t_i}}{\alpha}}\right)} = 0
\end{aligned}
\tag{7}
$$

Because there is not a closed form solution is not possible to solve (9), numerical techniques must be used. Software *R* provides the package *maxLik* to solve these equations. Hypotheses tests and confidence intervals for α, β, θ and λ can be obtained using the asymptotical normal distribution for $\hat{\alpha}$, $\hat{\beta}$, $\hat{\theta}$, and $\hat{\lambda}$, that is,

$$
\widehat{\Theta} \overset{a}{\sim} N\left\{\Theta, I_0^{-1}\right\},
\tag{8}
$$

where I_0 is the observed Fisher information matrix given by,

$$
I_0\left(\Theta\right) = \begin{pmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\theta} & I_{\alpha\lambda} \\ \cdot & I_{\beta\beta} & I_{\beta\theta} & I_{\beta\lambda} \\ \cdot & \cdot & I_{\theta\theta} & I_{\theta\lambda} \\ \cdot & \cdot & \cdot & I_{\lambda\lambda} \end{pmatrix}
$$

Carrasco *et al.* (2008) derive the observed Fisher information matrix for the parameters Θ of GMW distribution and the elements of matrix I_0 are given in Apendix.

3 Bayesian analysis

The Bayesian inference is an alternative to the maximum likelihood estimation of probability distributions.

In the Bayesian framework is necessary to specify a prior distribution for the unknown parameters. If the prior information is unavailable for a process, then, initial uncertainty about the parameters can be quantified with a noninformative prior distribution. We assume different prior distributions for the parameters α, β, θ and λ, denoted by $\pi\left(\alpha, \beta, \theta, \lambda\right)$. Thus the joint posterior distribution for the parameters is proportional to the product of the likelihood function (7) and the prior $\pi\left(\alpha, \beta, \theta, \lambda\right)$ that is,

$$p\left(\alpha, \beta, \theta, \lambda|t\right) \propto \pi\left(\alpha, \beta, \theta, \lambda\right) \frac{\beta^n}{\alpha^n} \prod_{i=1}^{n} \frac{t_i^{\theta-1}\left(\theta+\lambda\, t_i\right) \exp\left\{\lambda\, t_i - \frac{t_i^{\theta} \mathrm{e}^{\lambda\, t_i}}{\alpha}\right\}}{\left[1-\exp\left\{-\frac{t_i^{\theta} \mathrm{e}^{\lambda\, t_i}}{\alpha}\right\}\right]^{1-\beta}}. \tag{9}$$

A common specification of non informative prior considered is given by the product of independent gamma prior distributions given by

$$\pi\left(\alpha, \beta, \theta, \lambda\right) = G\left(\alpha, a_1, b_1\right) \times G\left(\beta, a_2, b_2\right) \times G\left(\theta, a_3, b_3\right) \times G\left(\lambda, a_4, b_4\right) \tag{10}$$

where $G\left(x, a, b\right) = \frac{b^a}{\Gamma(a)} x^{a-1} \mathrm{e}^{-bx}$, for $x > 0$ and $a, b > 0$.

The values of hyper-parameters a_i and b_i $(i = 1, 2, 3, 4)$ are choosing in order to provide ausence of information a priori. Generally the values are choosing as 0.01 or 0.1. An uniform prior for the parameters is also appropriated.

An another option for noninformative prior density is chose an improper prior assuming independence among the parameters given by,

$$\pi(\alpha, \beta, \lambda, \theta) \propto \frac{1}{\alpha\beta\lambda\theta}. \tag{11}$$

Let us denote the prior (13) as Jeffreys prior.

Since there is no closed form for marginal integration of we need to appeal for numerical procedures to extract characteristics of the marginal posterior distributions such as Bayesian estimations and credible intervals. We can then use MCMC algorithm (see for example, CASELLA and GEORGE, 1992) or the Metropolis-Hastings algorithm (see for example, GELFAND and SMITH, 1990; CHIB and GREENBERG, 1995) to obtain a sample of values of $\Theta = (\alpha, \beta, \theta, \lambda)$ from the joint posterior (11).

4 Simulated data

In this section we present and discuss the performance of MLE and Bayesian estimators, based on the point estimation, confidence intervals and coverage probabilities. The objective is to evaluate the effects of sample sizes and shapes of the hazard function in the estimation. To evaluate this effect, data set of size 10, 30, 50 and 100 are generated from the GMW distribution for different values of parameters corresponding to the several shapes of the hazard function. Marginal posterior distributions, Bayes estimators and credible intervals are obtained by using the MCMC algorithms. The chain is run for 15000 iterations with a burn-in period of 5000 and convergence monitored from the MCMC output and auto-correlation plots. The MCMC plots suggest we have achieved convergence with a rate of acceptance around $35\% - 50\%$. Also, the Raftery and Lewis (1992) diagnostics indicated convergence for all parameters. All reported results are based on 1000 simulation replications and they are shown in Tables 1-8. The simulation is conducted by using the statistical software package R (version 3.0.2). The set of values for the parameters were chosen in to order to provide the hazard function with increasing, decreasing, unimodaland bathtube shapes.

The density, reliability and hazard functions for each shape are displayed in Figures 1-4.

4.1 Increasing hazard function with parameters $\alpha = 1$, $\beta = 1$, $\theta = 1$ and $\lambda = 1$.

The density, reliable and razard functions of the GMW distribution are plotted in Figure 1 by considering the parameters $\alpha = 1$, $\beta = 1$, $\theta = 1$ e $\lambda = 1$.

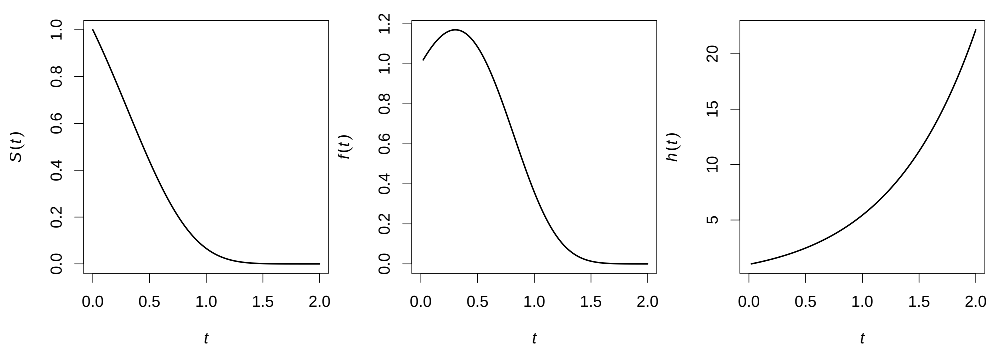


Figura 1 - Plot of reability, density and hazard functions of the GMW distribution for $\alpha = 1$, $\beta = 1$, $\theta = 1$ e $\lambda = 1$.

In Tables 1, 3, 5 and 7, the posterior means and standard deviance from each prior are compared with the MLE by considering several sample sizes.

A criterion for comparison of the quality of the prior distributions consists on checking the frequentists coverage probabilities of the posterior intervals arising from the priors. See the results in Table 2, 4, 6 and 8.

Tabela 1 - MLE, posterior estimates (mean) and standard deviation for Θ

	n	MLE	Uniform	Gamma	Jeffreys
$\alpha = 1$	10	1.2821 (1.7929)	2.3547 (0.8994)	0.9933 (0.1552)	1.1381 (0.3999)
	30	1.1352 (1.1451)	1.3918 (0.4775)	0.9953 (0.0709)	1.0349 (0.2186)
	50	1.0541 (0.8366)	1.1404 (0.2556)	0.9956 (0.0467)	1.0163 (0.1761)
	100	0.9870 (0.4965)	1.0758 (0.1796)	0.9980 (0.0295)	1.0103 (0.1199)
$\beta = 1$	10	2.0933 (1.4577)	1.0411 (0.7078)	1.1711 (0.3679)	1.1167 (0.3854)
	30	1.5750 (0.9736)	1.0544 (0.4356)	1.0755 (0.2170)	1.0357 (0.2084)
	50	1.4447 (0.9033)	1.0299 (0.2561)	1.0408 (0.1622)	1.0278 (0.1529)
	100	1.2874 (0.7164)	1.0102 (0.1813)	1.0200 (0.1169)	1.0131 (0.1148)
$\theta = 1$	10	1.4328 (1.4715)	2.0377 (0.9971)	1.1074 (0.3271)	1.3700 (0.3774)
	30	1.2011 (1.0130)	1.3446 (0.5059)	1.0334 (0.1961)	1.1479 (0.2167)
	50	1.1836 (0.8420)	1.1331 (0.2694)	1.0256 (0.1460)	1.0862 (0.1565)
	100	1.1250 (0.6579)	1.0759 (0.2007)	1.0155 (0.1123)	1.0425 (0.1119)
$\lambda = 1$	10	0.9171 (1.1231)	1.5400 (0.6371)	0.9984 (0.1156)	0.9046 (0.3389)
	30	0.9525 (0.6937)	1.1553 (0.3850)	1.0046 (0.0571)	0.9634 (0.2110)
	50	0.9160 (0.5789)	1.0648 (0.2464)	1.0005 (0.0440)	0.9692 (0.1757)
	100	0.9348 (0.4261)	1.0393 (0.1768)	1.0003 (0.0289)	0.9825 (0.1196)

Tabela 2 - Frequentist coverage probability of the 95% posterior intervals

	$\alpha = 1$				$\beta = 1$			
n	MLE	Uniform	Gamma	Jeffreys	MLE	Uniform	Gamma	Jeffreys
10	63.0%	94.4%	99.3%	99.0%	88.6%	97.2%	99.0%	99,2%
30	78.1%	95.5%	99.5%	99.0%	90.9%	97.2%	98.3%	99.2%
50	80.0%	96.9%	98.9%	99.2%	90.1%	97.9%	98.8%	99.8%
100	87.3%	97.3%	99.6%	99.1%	91.8%	97.3%	98.3%	99.6%
	$\theta = 1$				$\lambda = 1$			
n	MLE	Uniform	Gamma	Jeffreys	MLE	Uniform	Gamma	Jeffreys
10	88.3%	98.0%	99.4%	99.0%	90.9%	95.8%	99.7%	99.3%
30	85.8%	97.0%	98.9%	99.2%	92.8%	96.7%	99.7%	99.2%
50	85.5%	97.9%	99.1%	99.5%	91.9%	96.9%	99.3%	99.3%
100	85.1%	97.1%	98.8%	99.5%	95.0%	97.9%	99.1%	99.3%

4.2 Decreasing hazard function with parameters $\alpha = 3$, $\beta = 0.5$, $\theta = 1$ e $\lambda = 0.001$.

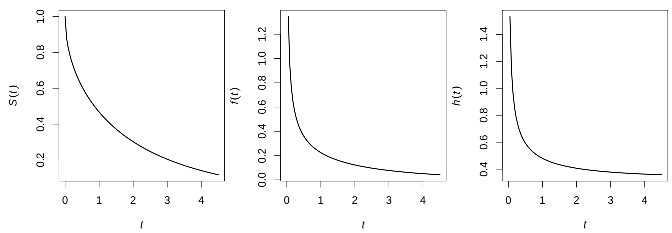


Figura 2 - Plot of reability, density and hazard functions of the GMW distribution for $\alpha = 3$, $\beta = 0.5$, $\theta = 1$ e $\lambda = 0.001$.

Tabela 3 - MLE, posterior estimates (mean) and standard deviation for Θ

	n	MLE	Uniform	Gamma	Jeffreys
$\alpha = 3$	10	3.5308 (4.7841)	6.7278 (2.6307)	3.5907 (1.7996)	3.1050 (0.8140)
	30	2.1906 (2.0883)	4.2732 (1.3872)	3.1315 (1.1766)	3.0643 (0.5000)
	50	2.2789 (1.4721)	3.7798 (1.0956)	3.0718 (0.9486)	3.0183 (0.3653)
	100	2.5394 (0.8741)	3.3495 (0.7314)	3.0162 (0.7137)	3.0103 (0.2567)
$\beta = 0.5$	10	2.3419 (2.3426)	0.5667 (0.2618)	0.7810 (0.3069)	0.6503 (0.2321)
	30	1.2208 (0.752)	0.5223 (0.1450)	0.6171 (0.1663)	0.5435 (0.1145)
	50	0.8263 (0.322)	0.5256 (0.1231)	0.5830 (0.1303)	0.5227 (0.0879)
	100	0.6134 (0.1285)	0.5150 (0.0851)	0.5428 (0.0882)	0.5076(0.0613)
$\theta = 1$	10	0.6565 (0.6557)	1.3454 (0.3753)	1.0205 (0.3108)	1.0689 (0.3030)
	30	0.6655 (0.3477)	1.1167 (0.2080)	0.9745 (0.1866)	1.0052 (0.1416)
	50	0.7629 (0.2868)	1.0701 (0.1628)	0.9814 (0.1555)	0.9994 (0.1055)
	100	0.8803 (0.1872)	1.0308 (0.1138)	0.9864 (0.1125)	0.9991 (0.0755)
$\lambda = 0.001$	10	0.1667 (0.2881)	0.0040 (0.0014)	0.0013 (0.001)	0.0010 (3e-04)
	30	0.0587 (0.0783)	0.0021 (6e-04)	0.0011 (6e-04)	0.0010 (1e-04)
	50	0.0396 (0.0543)	0.0016 (4e-04)	0.001 (4e-04)	9e-04 (1e-04)
	100	0.0202 (0.0373)	0.0013 (3e-04)	0.001 (3e-04)	9e-04 (1e-04)

Tabela 4 - Frequentist coverage probability of the 95% posterior intervals

	$\alpha = 1$				$\beta = 1$			
n	MLE	Uniform	Gamma	Jeffreys	MLE	Uniform	Gamma	Jeffreys
10	46.5%	96.7%	99.1%	99.6%	94.5 %	97.3 %	98.5 %	98.1%
30	57.0%	96.6%	97.1%	99.4%	93.3 %	97.3 %	97.2 %	98.3%
50	65.8%	97.2%	96.9%	99.7%	92.4 %	97.6 %	96.7 %	98.2%
100	79.8%	98.2%	97.7%	99.9%	92.7 %	97.0 %	97.5 %	98.7%
	$\theta = 1$				$\lambda = 1$			
n	MLE	Uniform	Gamma	Jeffreys	MLE	Uniform	Gamma	Jeffreys
10	71.5 %	95.4 %	98.9 %	98.8%	89.7 %	96.0 %	99.6 %	99.1%
30	77.1 %	96.0 %	97.0 %	99.0%	91.2 %	98.3 %	99.1 %	99.2%
50	82.9 %	96.2 %	95.8 %	98.6%	89.9 %	99.3 %	99.6 %	99.6%
100	89.8 %	96.6 %	96.9 %	98.8%	91.3 %	98.8 %	99.1 %	99.3%

4.3 Unimodal hazard function with parameters $\alpha = 0.2$, $\beta = 8$, $\theta = 0.5$ e $\lambda = 0.01$.

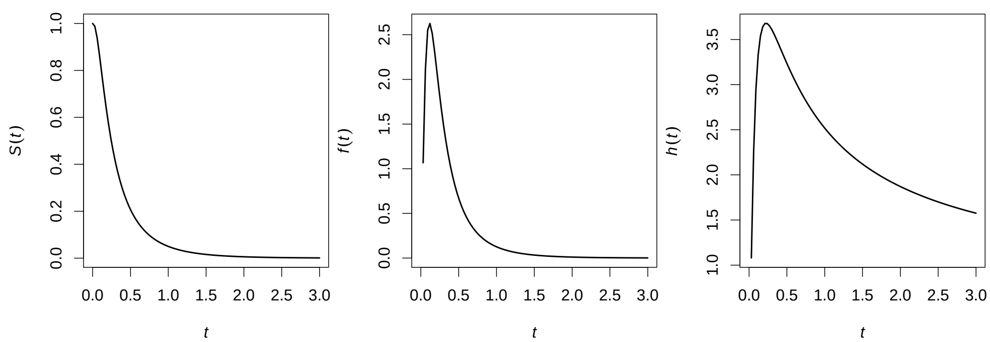


Figura 3 - Plot of reability, density and hazard functions of the GMW distribution for $\alpha = 0.2$, $\beta = 8$, $\theta = 0.5$ e $\lambda = 0.01$.

Tabela 5 - MLE, posterior estimates (mean) and standard deviation for Θ

	n	MLE	Uniform	Gamma	Jeffreys
$\alpha = 0.2$	10	0.2857 (0.2688)	0.1977 (0.0438)	0.1997 (0.0105)	0.2021 (0.0398)
	30	0.2112 (0.0589)	0.1990 (0.0250)	0.2005 (0.0138)	0.2014 (0.0224)
	50	0.2095 (0.0415)	0.2007 (0.0197)	0.2007 (0.0111)	0.2004 (0.0169)
	100	0.2058 (0.0287)	0.2003 (0.014)	0.2001 (0.0033)	0.1996 (0.0122)
$\beta = 8$	10	13.9437 (7.4299)	8.1209 (1.0217)	7.6961 (0.9715)	7.8964 (0.7179)
	30	13.6849 (5.765)	8.0709 (1.1183)	7.6314 (1.0545)	7.9573 (0.4082)
	50	11.1423 (3.7401)	8.0812 (1.1587)	7.7118 (1.0598)	7.9722 (0.3384)
	100	9.2869 (1.9756)	8.1085 (1.1691)	7.8814 (1.0798)	7.9808 (0.2293)
$\theta = 0.5$	10	0.4374 (0.2063)	0.5673 (0.1446)	0.5326 (0.0991)	0.5474 (0.1149)
	30	0.4221 (0.1325)	0.5307 (0.0758)	0.5297 (0.0617)	0.5123 (0.0618)
	50	0.4417 (0.0988)	0.5179 (0.0600)	0.5219 (0.0512)	0.5110 (0.0479)
	100	0.4655 (0.0648)	0.5125 (0.0441)	0.5117 (0.0353)	0.5051 (0.0335)
$\lambda = 0.01$	10	0.5518 (1.0267)	0.0429 (0.0149)	0.0146 (0.0232)	0.0095 (0.0091)
	30	0.194 (0.3063)	0.0238 (0.0083)	0.0125 (0.0115)	0.0102 (0.0071)
	50	0.1412 (0.2274)	0.0188 (0.0078)	0.0110 (0.0075)	0.0099 (0.0045)
	100	0.0792 (0.1385)	0.0144 (0.0029)	0.0103 (0.0037)	0.0101 (0.0034)

Tabela 6 - Frequentist coverage probability of the 95% posterior intervals

	$\alpha = 1$				$\beta = 1$			
n	MLE	Uniform	Gamma	Jeffreys	MLE	Uniform	Gamma	Jeffreys
10	92.7 %	93.7 %	99.5 %	96.0 %	94.9 %	99.4 %	99.3 %	99.6%
30	92.9 %	94.5 %	98.7 %	95.7 %	92.8 %	98.8 %	98.9 %	99.6%
50	93.5 %	95.2 %	98.4 %	96.9 %	98.2 %	99.1 %	99.2 %	99.3%
100	92.9 %	95.5 %	99.7 %	96.0 %	99.4 %	97.9 %	98.8 %	99.8%
	$\theta = 1$				$\lambda = 1$			
n	MLE	Uniform	Gamma	Jeffreys	MLE	Uniform	Gamma	Jeffreys
10	71.0 %	92.6 %	96.2 %	97.2 %	96.7 %	99.4 %	99.5 %	97.3%
30	78.7 %	95.1 %	97.4 %	96.6 %	97.1 %	99.8 %	99.4 %	98.7%
50	84.0 %	96.7 %	98.3 %	96.9 %	95.2 %	99.5 %	98.9 %	99%
100	94.7 %	96.2 %	98.1 %	98.1 %	93.7 %	99.5 %	99.4 %	99.4%

4.4 Bathtub hazard function with parameters $\alpha = 5$, $\beta = 0.5$, $\theta = 0.5$ e $\lambda = 0.5$.

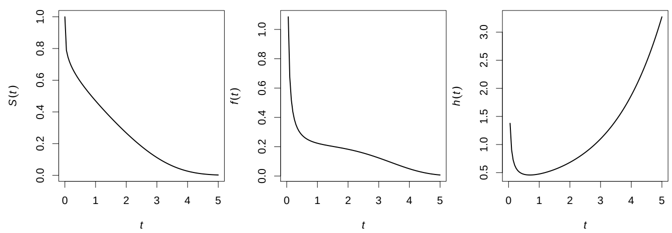


Figura 4 - Plot of reability, density and hazard functions of the GMW distribution for $\alpha = 5$, $\beta = 0.5$, $\theta = 0.5$ e $\lambda = 0.5$.

Tabela 7 - MLE, posterior estimates (mean) and standard deviation for Θ

	n	MLE	Uniform	Gamma	Jeffreys
$\alpha = 5$	10	11.2806 (7.3218)	8.2234 (2.2486)	4.8840 (1.7433)	5.0812 (0.8769)
	30	10.1307 (6.4406)	6.1489 (1.2049)	4.9114 (1.2029)	5.0252 (0.5633)
	50	7.2619 (3.3115)	5.3473 (0.6716)	4.8833 (0.5689)	5.0199 (0.4216)
	100	5.6768 (1.4708)	5.1652 (0.4921)	4.9627 (0.7363)	4.9940 (0.3057)
$\beta = 0.5$	10	0.5994 (0.6615)	0.5893 (0.2202)	0.6919 (0.2227)	0.6101 (0.2036)
	30	0.4783 (0.3317)	0.5358 (0.1297)	0.5763 (0.1211)	0.5387 (0.1165)
	50	0.4713 (0.1894)	0.5343 (0.1034)	0.5576 (0.1148)	0.5216 (0.0894)
	100	0.4746 (0.0971)	0.5143 (0.0632)	0.5271 (0.0677)	0.5070 (0.0606)
$\theta = 0.5$	10	1.0204 (0.7065)	0.7556 (0.3112)	0.6038 (0.2869)	0.6103 (0.2737)
	30	0.9045 (0.394)	0.5622 (0.1831)	0.5004 (0.1635)	0.5188 (0.1609)
	50	0.7795 (0.2668)	0.5300 (0.1604)	0.5005 (0.1607)	0.5120 (0.1306)
	100	0.7049 (0.1539)	0.4881 (0.0772)	0.4748 (0.0765)	0.4935 (0.0895)
$\lambda = 0.5$	10	0.5526 (0.3896)	0.5698 (0.2287)	0.4402 (0.1860)	0.4856 (0.1233)
	30	0.4886 (0.1965)	0.5255 (0.1199)	0.4831 (0.1086)	0.4954 (0.0787)
	50	0.4756 (0.1375)	0.5097 (0.0963)	0.4963 (0.0920)	0.5014 (0.0591)
	100	0.4525 (0.0899)	0.5138 (0.0574)	0.5019 (0.0364)	0.5001 (0.0427)

Tabela 8 - Frequentist coverage probability of the 95% posterior intervals

	$\alpha = 1$				$\beta = 1$			
n	MLE	Uniform	Gamma	Jeffreys	MLE	Uniform	Gamma	Jeffreys
10	38.8 %	97.2 %	98.7 %	99.8 %	62.2 %	97.6 %	97.2 %	98.9%
30	32.9 %	99.1 %	98.9 %	99.2 %	69.2 %	98.1 %	97.5 %	98.4%
50	46.9 %	99.0 %	98.7 %	99.6 %	73.1 %	96.8 %	95.7 %	98.4%
100	73.1 %	99.1 %	98.9 %	99.1 %	83.9 %	98.0 %	98.5 %	98.8%
	$\theta = 1$				$\lambda = 1$			
n	MLE	Uniform	Gamma	Jeffreys	MLE	Uniform	Gamma	Jeffreys
10	64.8 %	97.4 %	97.6 %	98.9 %	79.4 %	95.4 %	95.0 %	99.1%
30	66.7 %	97.2 %	96.6 %	97.6 %	79.2 %	95.9 %	96.5 %	98.3%
50	80.2 %	95.6 %	94.1 %	97.1 %	87.0 %	94.0 %	94.2 %	98.7%
100	87.3 %	97.9 %	96.9 %	97.3 %	90.8 %	96.4 %	98.4 %	98.5%

5 Discussion

Some of the points are quite clear from the numerical results. It is observed that the Bayesian analysis performs better than the MLE for the parameters α, β, θ and λ in all class of hazard function of GMW distributon. In the comparison among the three prior distributions in study we observe that Jeffreys prior provides better estimation than both uniform and gamma priors. By considering the parameter theta, Jeffreys and gamma prior distributions performs quite similar.

In the Bayesian approach note that uniform prior does not provide good estimates among the three considered class of priors. In terms of frequentist coverage probability, Tables illustrate that the coverage probabilities for maximum likelihood estimation are far below the nominal level. On the other hand, the coverage probabilities are higher than the nominal level for the Bayesian procedure. Our simulation study also shows that the Jeffreys prior provides coverage probabilities often close to 99% level. With these intervals we can be sure that the intervals, on average, will have at least the desired coverage probability regardless of score levels. Therefore, based on this simulated study we recommend the Bayesian approach with Jeffreys prior as the best inference to estimate the parameters of GMW distribution.

As expected it is observed that the performances of all estimates of α, β, θ and λ become better and closer when the sample size increases.

6 An example with literature data

Let us consider the data set related to the lifetime of lamps introduced in Aarset (1985). Mudholkar and Srivastava (1993) illustrates the fit of the Exponentiated Weibull distribution. The data is shown in Table 9.

Tabela 9 - Lifetime of 50 electronic devices

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

For the Bayesian analysis of the data let us assume the Jeffreys , independent uniform and also Gamma priors (with hyperparameter values a=0.01 and b=0.01) for each parameter. Using the software R, we performed a MCMC simulation with 405000 iterations and discarded the first 5000 as a burn-in. We have used the software CODA for monitoring the behavior of the chains. The Raftery-Lewis diagnostic has detected convergence of the MCMC and the algorithm showed a rate of acceptance around 35% and 50%. The maximum likelihood estimators are also evaluated. The results for both distributions are shown in Tables 10-13. We plot the posterior densities obtained from the three priors for the parameters Θ in Figure 5.

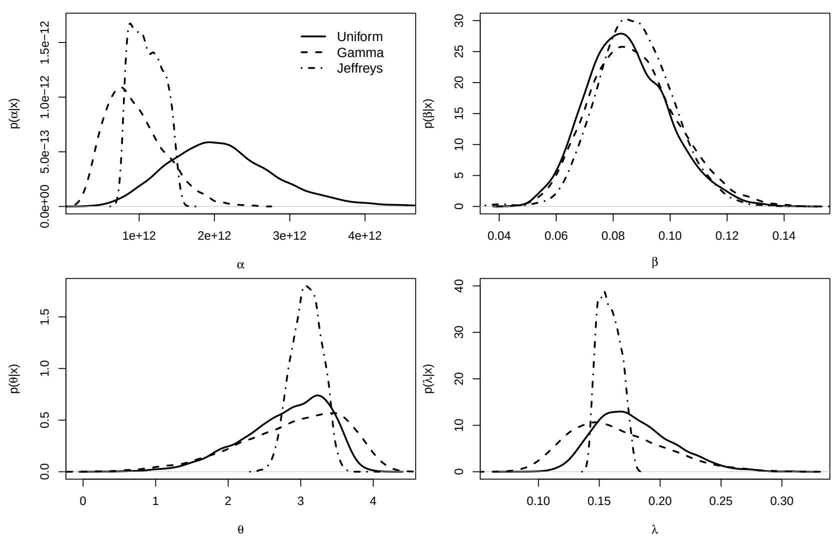


Figura 5 - Marginal density of the parameters α, β, θ and λ.

The tables 10-13 show the posterior means, standard deviation and 95% confidence posterior intervals for the parameters α, β, θ and λ by considering Jeffreys, uniform and gamma priors. The MLE are also computed.

Tabela 10 - Posterior summaries for α

Priors	$\widehat{\alpha}$	standard-deviation	95% interval
Uniform	2.1×10^{12}	7.4×10^{11}	$\left(9.1 \times 10^{11}, 3.7 \times 10^{12}\right)$
Gamma	9.7×10^{11}	3.8×10^{11}	$\left(3.6 \times 10^{11}, 1.8 \times 10^{12}\right)$
Jeffreys	1.1×10^{12}	2.0×10^{11}	$\left(7.9 \times 10^{11}, 1.8 \times 10^{12}\right)$
MLE	6.5×10^{11}	2.9×10^{11}	$\left(6.9 \times 10^{10}, 1.2 \times 10^{12}\right)$

Tabela 11 - Posterior summaries for β

Priors	$\widehat{\beta}$	standard-deviation	95% interval
Uniform	0.0851	0.0144	$(0.0595, 0.1165)$
Gamma	0.0874	0.0153	$(0.0606, 0.1199)$
Jeffreys	0.0878	0.0137	$(0.0639, 0.1145)$
MLE	0.0931	0.0136	$(0.0664, 0.1198)$

Tabela 12 - Posterior summaries for θ

Priors	$\widehat{\theta}$	standard-deviation	95% interval
Uniform	2.7992	0.5929	$(1.4631, 3.6515)$
Gamma	2.8916	0.7335	$(1.1908, 3.9815)$
Jeffreys	3.0826	0.2036	$(2.6826, 3.4457)$
MLE	3.1245	0.3440	$(2.4503, 3.7987)$

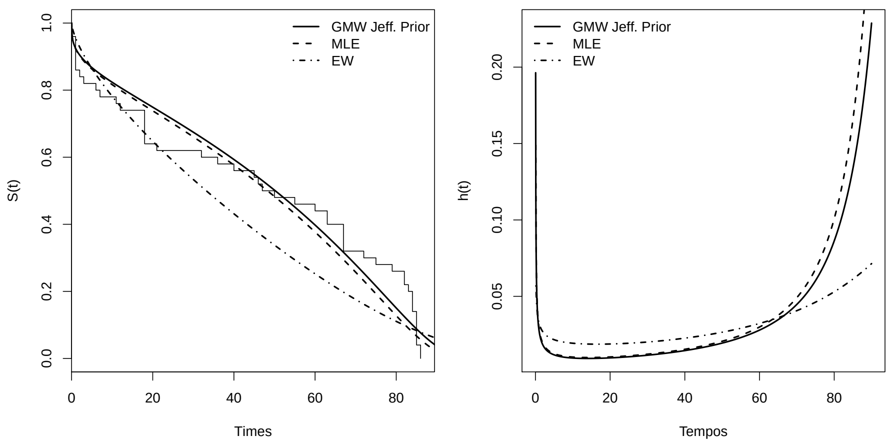
Tabela 13 - Posterior summaries for λ

Priors	$\widehat{\lambda}$	standard-deviation	95% interval
Uniform	0.1785	0.0319	$(0.1277, 0.2484)$
Gamma	0.1644	0.0392	$(0.1028, 0.2531)$
Jeffreys	0.1583	0.0088	$(0.1438, 0.1753)$
MLE	0.1515	0.0199	$(0.1125, 0.1906)$

We have also compared the fitting of the GMW and Exponentiated Weibull distributions. Table 14 provides the DIC values corresponding to the each distributions for different priors. Thus, we conclude that the GMW distribution yields better fit to the data set, since its DIC value is smaller. The graphical comparison of the empirical distribution function and the fitted distribution functions is given in Figure 6. The hazard functions are also plotted for comparison in Figure 6b. Figure 6 suggests that the GMW distribution is more appropriate than the Exponentiated Weibull distribution for this data set.

Tabela 14 - Discrimination criteria DIC (Deviance information criterion)

Distribution	DIC
GMW with Uniform Prior	443.4
GMW with Gamma Prior	445.2
GMW with Jeffreys Prior	443.0
EW with Gamma Prior	463.6



(a) (b)

Figura 6 - Plot of reliable (a) and hazard (b) functions for Exponentiated Weibull, GMW and Empirical distribuitions.

7 Presence of covariates

Survival data from 30 patients with AML is given in Table 15 (LEE, 1992). Age is a continuous covariate.

Tabela 15 - Survival times vs. Age of 30 AML patients

Survival Time (t)	Age (X)	Survival Time(t)	Age (X)
18	35	8	72
9	42	2	60
28	33	26	56
31	20	10	61
39	22	4	59
19	45	3	69
45	37	4	70
6	19	18	54
8	44	8	74
15	26	3	53
23	48	14	66
28	32	3	64
7	21	13	54
12	51	13	60
9	65	35	68

To analyze this data, we assume the regression model in the presence of covariate age.

Let us be the modified generalized Weibull density (1) to the survival times with scale parameter given by,

$$\alpha(x_i) = exp(\beta_0 + \beta_1 x_i) \tag{12}$$

for $i = 1, 2, \ldots, n$ (number of individuals), and X which denotes a covariate age.

We consider the Bayesian analysis assuming the following prior distributions for the parameters of the model

$$\beta \sim \Gamma(0.01, 0.01), \lambda \sim \Gamma(0.01, 0.01), \theta \sim \Gamma(0.01, 0.01), \beta_0 \sim N(0, 100), \beta_1 \sim N(0, 100). \tag{13}$$

The marginal posteriors of interest are based on 50000 samples simulated from the joint distribution with burn-in= 5000 samples.

Tables 16-20 give the maximum likelihood, Bayesian summaries and intervals for the parameters.

Tabela 16 - Point estimators and 95% intervals for parameter β (data with covariate)

Methods	Estimator	s.d.	95% CI
MLE	9.4166	2.2045	(5.0957 ; 13.7376)
Gamma	8.2594	2.5224	(4.5007; 14.5373)
Jeffreys	8.2694	2.5249	(4.4055; 13.9446)

Tabela 17 - Point estimators and 95% intervals for parameter θ (data with covariate)

Methods	Estimator	s.d.	95% CI
MLE	0.3669	0.0470	(0.2746 ; 0.4591)
Gamma	0.3292	0.0447	(0.2446; 0.4203)
Jeffreys	0.3309	0.0464	(0.2461; 0.4255)

Tabela 18 - Point estimators and 95% intervals for parameter λ (data with covariate)

Methods	Estimator	s.d.	95% CI
MLE	0.0206	0.0034	(0.0138 ; 0.0273)
Gamma	0.0276	0.0042	(0.0198; 0.0367)
Jeffreys	0.0277	0.0044	(0.0198; 0.0367)

Tabela 19 - Point estimators and 95% intervals for parameter β_0 (data with covariate)

Methods	Estimator	s.d.	95% CI
MLE	0.7546	0.2574	(0.2500; 1.2591)
Gamma	0.8487	0.2783	(0.3319; 1.3995)
Jeffreys	0.8553	0.2752	(0.2938; 1.3760)

Tabela 20 - Point estimators and 95% intervals for parameter β_1 (data with covariate)

Methods	Estimator	s.d.	95% CI
MLE	-0.0112	0.0041	(-0.0194 ; -0.0030)
Gamma	-0.0117	0.0045	(-0.0205; -0.0032)
Jeffreys	-0.0117	0.0044	(-0.0205; -0.0032)

From the results of Tables 16-20, we observed that the covariate age has a small effect on survival times, because zero is not included in the 95% credible intervals. Also, note that Jeffreys and gamma priors present quite similar estimation for all parameters. We plot the posterior density functions obtained by MCMC method in Figure 7. The plots show the same performance of the Jeffreys and gamma priors applied to analyse these data.

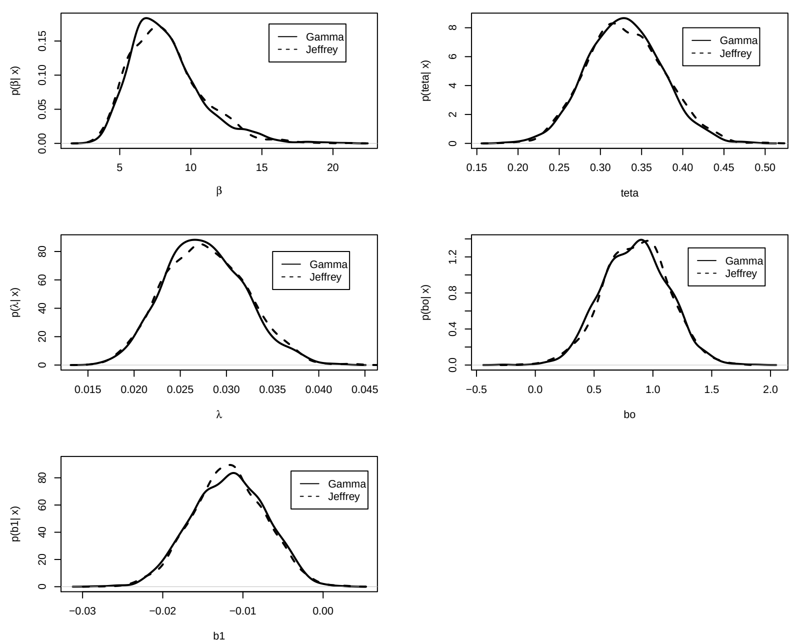


Figura 7 - Marginal density of the parameters β, θ, λ, β_0 and β_1.

8 Conclusion

We observed that the generalized modified Weibull distribution can be used quite effectively in analyze of lifetime data. It includes important distributions as special cases and exhibits decreasing, increasing, unimodal and bathtub hazard rate depending on its parameters. In this paper we considered the Bayesian estimation of the unknown parameters of the GMW distribution by assuming some noninformative priors. After an extensive Monte Carlo simulations we concluded that the Bayesian estimation performs better than the maximum likelihood for any shape of the hazard functions. In terms of frequentist coverage probability our simulation study indicates that the Jeffreys prior have performed better than the other priors for the all the parameters.

NIIYAMA, C. A.; MOALA, F. A.; OIKAWA, S. M. Análise bayesiana de distribuição Weibull modificada generalizada. *Rev. Bras. Biom.,* Lavras, v.34, n.4, p.575-596, 2016.

- RESUMO: Recentemente, Carrasco *et al.* (2008) propuseram a distribuição Weibull generalizada modificada (GMW). Esta distribuição apresenta a função de risco com formas de banheira, unimodal e monotônica. Além disso, a GMW inclui A Weibull, o valor extremo, A Weibull exponencial e as distribuições generalizadas de Rayleigh como casos especiais. Neste artigo, consideramos uma análise bayesiana completa, assumindo priores não informativos para os parâmetros desconhecidos e a análise bayesiana é comparada à abordagem de máxima verossimilhança. Introduzimos os algoritmos MCMC para gerar amostras a partir das distribuições posteriores, a fim de avaliar os estimadores e os intervalos. A ilustração numérica da metodologia proposta é baseada em dados simulados e em dois conjuntos de dados de vida útil.

- PALAVRAS-CHAVE: Weibull modificada generalizada; estimativa bayesiana; priores não informativas; métodos MCMC; máxima verossimilhança; priores de Jeffrey; covariável.

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9 Appendix

Observed Fisher information matrix

In this Appendix we reproduce the comutation of Fisher information matrix $I(\Theta)$ presented by Carrasco et al. (2008) as follow:

$$I(\Theta) = \begin{pmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\theta} & I_{\alpha\lambda} \\ \cdot & I_{\beta\beta} & I_{\beta\theta} & I_{\beta\lambda} \\ \cdot & \cdot & I_{\theta\theta} & I_{\theta\lambda} \\ \cdot & \cdot & \cdot & I_{\lambda\lambda} \end{pmatrix}$$

where the elements are of observed information matrix given by:

$$\begin{aligned}
I_{\alpha\alpha} &= \sum_{i=1}^{n} 2\,\frac{\beta\, {t_i}^{\theta} y_i}{\alpha^3 w_i} - \frac{\beta\, {t_i}^{2\,\theta} \mathrm{e}^{2\,\lambda\, t_i - u_i}}{\alpha^4 w_i} - \frac{\beta\, {t_i}^{2\,\theta} {y_i}^2}{\alpha^4 {w_i}^2} \\
&+ \sum_{i=1}^{n} \alpha^{-2} - 2\,\frac{{t_i}^{\theta} z_i}{\alpha^3 v_i} - \frac{{t_i}^{2\,\theta} \mathrm{e}^{2\,\lambda\, t_i + u_i}}{\alpha^4 v_i} + \frac{{t_i}^{2\,\theta} {z_i}^2}{\alpha^4 {v_i}^2} \\
I_{\alpha\beta} &= \sum_{i=1}^{n} -\frac{{t_i}^{\theta} y_i}{\alpha^2 w_i} \\
I_{\alpha\theta} &= \sum_{i=1}^{n} \frac{\mathrm{e}^{\lambda\, t_i} \log(t_i) \left[\left(y_i + z_i - 2\,\mathrm{e}^{\lambda\, t_i}\right)(\beta - 1) {t_i}^{2\,\theta}\right]}{\alpha^3 {w_i}^2 {v_i}^2} \\
&- \sum_{i=1}^{n} \frac{\mathrm{e}^{\lambda\, t_i} \log(t_i) {t_i}^{\theta} \left[(1 + 3\,\beta)\mathrm{e}^{-u_i} - \mathrm{e}^{-2\,u_i}\beta - \mathrm{e}^{2\,u_i} + (3 + \beta)\mathrm{e}^{u_i} - 3\,\beta - 3\right]}{\alpha^2 {w_i}^2 {v_i}^2} \\
I_{\alpha\lambda} &= \sum_{i=1}^{n} \frac{\mathrm{e}^{\lambda\, t_i} t_i \left[\left(y_i + z_i - 2\,\mathrm{e}^{\lambda\, t_i}\right)(\beta - 1) {t_i}^{2\,\theta}\right]}{\alpha^3 {w_i}^2 {v_i}^2} \\
&- \sum_{i=1}^{n} \frac{\mathrm{e}^{\lambda\, t_i} {t_i}^{\theta+1} \left[(1 + 3\,\beta)\mathrm{e}^{-u_i} - \mathrm{e}^{-2\,u_i}\beta - \mathrm{e}^{2\,u_i} + (3 + \beta)\mathrm{e}^{u_i} - 3\,\beta - 3\right]}{\alpha^2 {w_i}^2 {v_i}^2} \\
I_{\beta\beta} &= -\frac{n}{\beta^2} \\
I_{\beta\theta} &= \sum_{i=1}^{n} \frac{{t_i}^{\theta} \log(t_i) y_i}{\alpha\, w_i} \\
I_{\beta\lambda} &= \sum_{i=1}^{n} \frac{{t_i}^{\theta+1} y_i}{\alpha\, w_i} \\
I_{\theta\theta} &= \sum_{i=1}^{n} \frac{\beta\, {t_i}^{\theta} (\log(t_i))^2 y_i}{\alpha\, w_i} - \frac{\beta\, {t_i}^{2\,\theta} (\log(t_i))^2 \mathrm{e}^{2\,\lambda\, t_i - u_i}}{\alpha^2 w_i} - \frac{\beta\, {t_i}^{2\,\theta} (\log(t_i))^2 {y_i}^2}{\alpha^2 {w_i}^2} \\
&+ \sum_{i=1}^{n} \frac{(2 + (\theta + \lambda\, t_i) \log(t_i)) \log(t_i)}{\theta + \lambda\, t_i} - \frac{\left({t_i}^{\theta-1} \log(t_i)\theta + {t_i}^{\theta-1} + {t_i}^{\theta} \log(t_i)\lambda\right)^2}{({t_i}^{\theta-1}\theta + {t_i}^{\theta}\lambda)^2} \\
&+ \sum_{i=1}^{n} \frac{{t_i}^{2\,\theta} (\log(t_i))^2 {z_i}^2}{\alpha^2 {v_i}^2} - \frac{{t_i}^{\theta} (\log(t_i))^2 z_i}{\alpha\, v_i} - \frac{{t_i}^{2\,\theta} (\log(t_i))^2 \mathrm{e}^{2\,\lambda\, t_i + u_i}}{\alpha^2 v_i} \\
I_{\theta\lambda} &= \sum_{i=1}^{n} \frac{\beta\, {t_i}^{\theta+1} \log(t_i) y_i}{\alpha\, w_i} - \frac{\beta\, {t_i}^{2\,\theta+1} \mathrm{e}^{2\,\lambda\, t_i - u_i} \log(t_i)}{\alpha^2 w_i} - \frac{\beta\, {t_i}^{2\,\theta+1} {y_i}^2 \log(t_i)}{\alpha^2 {w_i}^2} \\
&+ \sum_{i=1}^{n} \frac{{t_i}^{\theta} \log(t_i)}{{t_i}^{\theta-1}\theta + {t_i}^{\theta}\lambda} - \frac{t_i \theta \log(t_i)}{(\theta + \lambda\, t_i)^2} - \frac{t_i}{(\theta + \lambda\, t_i)^2} - \frac{{t_i}^2 \log(t_i)\lambda}{(\theta + \lambda\, t_i)^2} \\
&+ \sum_{i=1}^{n} \frac{{t_i}^{2\,\theta+1} {z_i}^2 \log(t_i)}{\alpha^2 {v_i}^2} - \frac{{t_i}^{\theta+1} \log(t_i) z_i}{\alpha\, v_i} - \frac{{t_i}^{2\,\theta+1} \mathrm{e}^{2\,\lambda\, t_i + u_i} \log(t_i)}{\alpha^2 v_i}
\end{aligned}$$

$$I_{\lambda\lambda} = \sum_{i=1}^{n} \frac{\beta\, t_i{}^{\theta+2} y_i}{\alpha\, w_i} - \frac{\beta\, t_i{}^{2\,\theta+2} \mathrm{e}^{2\,\lambda\, t_i - u_i}}{\alpha^2 w_i} - \frac{\beta\, t_i{}^{2\,\theta+2} y_i{}^2}{\alpha^2 w_i{}^2} - \frac{t_i{}^{2\,\theta}}{(t_i{}^{\theta-1}\theta + t_i{}^{\theta}\lambda)^2}$$
$$+ \sum_{i=1}^{n} \frac{t_i{}^{2\,\theta+2} z_i{}^2}{\alpha^2 v_i{}^2} - \frac{t_i{}^{\theta+2} z_i}{\alpha\, v_i} - \frac{t_i{}^{2\,\theta+2} \mathrm{e}^{2\,\lambda\, t_i + u_i}}{\alpha^2 v_i}$$

onde $u_i = \left[\frac{t_i^{\theta}\exp\{\lambda t_i\}}{a}\right]$, $v_i = \exp\left\{\frac{t_i^{\theta}\exp\{\lambda\, t_i\}}{\alpha}\right\} - 1$, $w_i = 1 - \exp\left\{-\frac{t_i^{\theta}\exp\{\lambda\, t_i\}}{\alpha}\right\}$, $x_i = -1 + \left(1 - \exp\left\{-\frac{t_i^{\theta}\exp\{\lambda\, t_i\}}{\alpha}\right\}\right)^{\beta}$, $y_i = \exp\left\{\lambda t_i - \frac{t_i^{\theta}\exp\{\lambda\, t_i\}}{\alpha}\right\}$ e $z_i = \exp\left\{\lambda t_i + \frac{t_i^{\theta}\exp\{\lambda\, t_i\}}{\alpha}\right\}$.

This way, the inverse of observed information matrix given by:

$$I^{-1}(\widehat{\Theta}) = \begin{pmatrix} var(\widehat{\alpha}) & cov(\widehat{\alpha},\widehat{\beta}) & cov(\widehat{\alpha},\widehat{\theta}) & cov(\widehat{\alpha},\widehat{\lambda}) \\ cov(\widehat{\beta},\widehat{\alpha}) & var(\widehat{\beta}) & cov(\widehat{\beta},\widehat{\beta}) & cov(\widehat{\beta},\widehat{\lambda}) \\ cov(\widehat{\theta},\widehat{\alpha}) & cov(\widehat{\theta},\widehat{\beta}) & var(\widehat{\theta}) & cov(\widehat{\theta},\widehat{\lambda}) \\ cov(\widehat{\lambda},\widehat{\alpha}) & cov(\widehat{\lambda},\widehat{\beta}) & cov(\widehat{\lambda},\widehat{\beta}) & var(\widehat{\lambda}) \end{pmatrix} \tag{14}$$