THE BETA EXPONENTIATED WEIBULL GEOMETRIC DISTRIBUTION: MODELING, STRUCTURAL PROPERTIES, ESTIMATION AND AN APPLICATION TO A CERVICAL INTRAEPITHELIAL NEOPLASIA DATASET

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▪ ABSTRACT: A new distribution, the so called beta exponentiated Weibull geometric (BEWG) distribution is proposed. The new distribution is generated from the logit of a beta random variable and includes the exponentiated Weibull geometric distribution as particular case. Various structural properties including explicit expressions for the moments, moment generating function, mean deviation of the new distribution are derived. The estimation of the model parameters is performed by maximum likelihood method. The usefulness of the model was showed by using a real dataset. In order to validate the results a simulation bootstrap is presented in this paper.

▪ KEYWORDS: Maximum likelihood estimation; moment generating function; reliability function.

1 Introduction and motivation

Let $G(x;\phi)$ be the cumulative distribution function (cdf) of an absolutely continuous random variable following the distribution G, where $\phi \in \Omega$ is the parameter vector. A general class generated from the logit of a beta random variable

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has been introduced by (EUGENE; LEE; FAMOYE, 2002) with the cumulative distribution function (cdf) of the Beta-G distribution having the form

$$\begin{aligned} F(x;a,b,\phi) &= \frac{1}{B(a,b)}\int_0^{G(x;\phi)} w^{a-1}(1-w)^{b-1}dw \\ &= \frac{B_{G(x;\phi)}(a,b)}{B(a,b)} = I_{G(x;\phi)}(a,b), \end{aligned} \tag{1}$$

where $a > 0$ and $b > 0$ are two additional parameters whose role is to introduce skewness and to vary tail weight , and $G(x;\phi)$ is an arbitrary parent / baseline cdf of a random variable, $B_y(a,b) = \int_0^y w^{a-1}(1-w)^{b-1}dw$ is the incomplete beta function with $B(a,b) = B_1(a,b)$ and $I_y(a,b) = B_y(a,b)/B(a,b)$ is the incomplete beta function ratio.

One major benefit of this class of distributions is its ability of fitting skewed data that cannot be properly fitted by existing distributions. If $b = 1$, $F(x) = G(x)^a$ and then F is usually called the exponentiated G distribution (or the Lehmann type-I distribution). See, for example, the exponentiated Weibull in (MUDHOLKAR; SRIVASTAVA; FREIMER, 1995) and exponentiated exponential distributions in (GUPTA; KUNDU, 1995).

Indeed, if Z is a beta distributed random variable with parameters a and b, then the cdf of $X = G^{-1}(Z)$ agrees with the cdf given in (1). As customary, a random variable X with the cdf (1) is said to have a beta-G (BG) distribution and will be denoted by $X \sim \mathrm{BG}(a,b,\phi)$. Some special cases of BG distributions are given below:

(i) If $G(x;\phi)$ is the cdf of a standard uniform distribution, then the cdf given in Equation (1) yields the cdf of a beta distribution with parameters a and b.

(ii) If a is an integer value and $b = n - a + 1$, then the cdf (1) becomes

$$\begin{aligned} F(x;a,b,\phi) &= \frac{1}{B\left(a,n-a+1\right)} \\ &\quad \times \int_0^{G(x;\phi)} w^{a-1}(1-w)^{b-1}dw \\ &= \sum_{i=a}^{n} \tbinom{n}{i} \left[G(x;\phi)\right]^i \left[1-G(x;\phi)\right]^{n-i}, \end{aligned}$$

which is really the cdf of the a_{th} order statistic of a random sample of size n from distribution $G(x;\phi)$.

(iii) If $a = b = 1$, then the cdf (1) reduces to $F(x;\phi) = G(x;\phi)$.

(iv) If $a = 1$, then the cdf (1) reduces to $F(x;b,\phi) = \left[1 - G(x;\phi)\right]^b$.

(v) If $b = 1$, then the cdf (1) reduces to $F(x;a,\phi) = \left[G(x;\phi)\right]^a$.

The two classes given in (iv) and (v) are called, respectively, the frailty parameter and resilience parameter families with underlying distribution $G(x;\phi)$ (see (MARSHALL; OLKIN, 2007)). Clearly, for positive integer values of $b(a)$, the cdf given in (iv)((v)) above is the cdf of a series (parallel) system with $b(a)$ independent components all having the cdf $G(x;\phi)$. Some well-known distributions belonging to the resilience parameter family are the exponentiated Weibull (EW) distribution see; (MUDHOLKAR; SRIVASTAVA; FREIMER, 1995), the generalized (or exponentiated) exponential distribution proposed by (GUPTA; KUNDU, 1999), the exponentiated type distributions introduced by (NADARAJAH; KOTZ, 2006). The generalized exponential-geometric (GEG) distribution of (SILVA; ORTEGA; CORDEIRO, 2010) also belongs to the resilience parameter family.

For general a and b, we can express the cdf given in (1) in terms of the well-known hypergeometric function defined by

$$F(x;a,b,\phi) = \frac{G(x;\phi)^a}{aB(a,b)} \quad {}_2F_1(a, 1-b, a+1; G(x;\phi)), \tag{2}$$

where

$${}_2F_1(\alpha,\theta,\gamma;x) = \sum_{i=0}^{\infty} \frac{(\alpha)_i(\theta)_i}{(\gamma)_i \quad i!} x^i, |x| < 1,$$

where $(\alpha)_i = \Gamma(\alpha+i)/\Gamma(\alpha) = \alpha(\alpha+1)...(\alpha+i-1)$ denotes the ascending factorial of α. In principle, We can obtain the properties of $F(x)$ for any BG distribution, defined from a parent $G(x;\phi)$ in (1), following from the properties of the hypergeometric function which are well established in the literature; see, for example, Section 9.1 of (GRADSHTEYN; RYZHIK, 2000). The probability density function (pdf) and hazard (failure) rate functions of a BG distribution corresponding to (1) are given by

$$f(x;a,b,\phi) = \frac{g(x;\phi)}{B(a,b)} G(x;\phi)^{a-1} \left\{1 - G(x;\phi)\right\}^{b-1}, \tag{3}$$

and

$$h(x;a,b,\phi) = \frac{g(x;\phi)G(x;\phi)^{a-1}\left\{1-G(x;\phi)\right\}^{b-1}}{B(a,b)I_{[1-G(x;\phi)]}(a,b)},$$

respectively, where $I_{[1-G(x;\phi)]}(a,b) = 1 - I_{G(x;\phi)}(a,b) = \overline{F}(x;a,b,\phi)$ is the survival function of a BG distribution corresponding to the cdf (1).

The Beta Generalized (Beta-G), introduced by (SINGH et al., 1988) is a rich class of generalized distributions. This class has captured a considerable attention over the last few years. (SEPANSKI; KONG, 2007) applied the Beta-G distribution to model the size distribution of income. Moreover, this distribution has been studied in literature for various forms of G. The distributions that have been explored are the Beta Normal (BN) (EUGENE; LEE; FAMOYE, 2002), the Beta Fréchet (BFr) distribution (NADARAJAH; KOTZ, 2004), the Beta

Exponential (BE) distribution (NADARAJAH; KOTZ, 2006), the Beta Weibull (BW) distribution (LEE; FAMOYE; OLUMOLADE, 2007), and (CORDEIRO; SILVA; ORTEGA, 2011) present beta Weibull geometric. (BARRETO-SOUZA; S.; CORDEIRO, 2010) introduced the Beta Generalized Exponential (BGE) distribution and the Beta Modified Weibull (BMW) distribution was discussed by (SILVA; ORTEGA; CORDEIRO, 2010). Beta Inverse Weibull (BIW), (KHAN; KING, 2013), are some new extensions of the Beta G class of distributions. Recently, (CORDEIRO; SILVA; ORTEGA, 2011) introduced the beta-Weibull geometric distribution with $G(x;\phi)$ in (1) to be the cdf of the Weibull-geometric distribution of (BARRETO-SOUZA; MORAIS; CORDEIRO, 2011). Other generalizations can be seem in (WANG; ELBATAL, 2015; NOFAL et al., 2018; MEROVCI; ELBATAL, 2015). proposed new distribution, generated from the logit of a beta random variable, which extends the exponential-geometric distribution of (ADAMIDIS; LOUKAS, 1998) called the beta exponential-geometric distribution. Besides, (BIDRAM; BEHBOODIAN; TOWHIDI, 2013) introduced new distribution generated from the logit of a beta random variable and includes the Weibull geometric distribution of (BARRETO-SOUZA; MORAIS; CORDEIRO, 2011).

In this paper, we attempt to generalized the exponentiated Weibull geometric (EWG) distribution of Mahmoudi and Shiran (2012) by taking $G(x;\phi)$ in (1) to the cdf of a (EWG) distribution. (MAHMOUDI; SHIRMAN, 2012) compounded an exponentiated Weibull distribution with a geometric distribution and obtained a new lifetime distribution with hazard rate function can be decreasing, increasing, bathtube and unimodal. Such bathtub hazard curves have nearly flat middle portions and the corresponding densities have a positive anti-mode. Unimodal hazard rates can be observed in course of a disease whose mortality reaches a peak after some finite period and then declines gradually, also, it is well motivated for industrial applications and biological studies.

The reminder of the paper is organized as follows. In Section 2, we define the beta exponentiated Weibull geometric (BEWG) distribution. The expansion for the cumulative and density functions of the (BEWG) distribution and some special cases are proposed in Section 2. Moments, moment generating function and mean deviation are discussed in Section 3. In Section 4 we obtain the Rényi and Shannon entropy of BEWG distribution. Maximum likelihood estimation is presented in Section 5. In Section 6 we verify the usefulness of the proposed model by fitting it to a real dataset on cervical intraepithelial neoplasia, as well as, present the results of a simulation study designed to asses the frequentist properties of the maximum likelihood estimators as well as the coverage probabilities of the confidence intervals. Some conclusions and final remarks are presented in Section 7.

2 Beta Exponentiated Weibull Geometric distribution

Suppose that $\{Y_i\}_{i=1}^{Z}$ are independent and identically distributed (iid) random variables following an exponentiated Weibull distribution $\mathrm{EW}(\alpha,\beta,\theta)$ with cdf given by

$$F(x;\alpha,\beta,\theta)=\left(1-e^{-(\alpha x)^{\beta}}\right)^{\theta},x>0,$$

and N a discrete random variable having a geometric distribution with probability function $P(n;p)=(1-p)p^{n-1}$ for $n\in\mathbb{N}$ and $p\in(0,1)$. Let $X_{(1)}=\min\{X_i\}_{i=1}^{N}$, the conditional cumulative distribution of $X_{(1)}\mid N=n$ is given by

$$\begin{aligned}G_{X_{(1)}|N=n}(x) &= 1-[1-F(x)]^{n}\\ &= 1-\left[1-\left(1-e^{-(\alpha x)^{\beta}}\right)^{\theta}\right]^{n},\end{aligned}$$

the cumulative distribution function of $X_{(1)}$ is given by

$$\begin{aligned}G(x;\alpha,\beta,\theta,p) &= (1-p)\sum_{n=1}^{\infty}p^{n-1} \\ &\quad\times\left\{1-\left[1-\left(1-e^{-(\alpha x)^{\beta}}\right)^{\theta}\right]^{n}\right\} \qquad (4)\\ &= \frac{1-\left(1-e^{-(\alpha x)^{\beta}}\right)^{\theta}}{1-p\left(1-e^{-(\alpha x)^{\beta}}\right)^{\theta}}. \qquad (5)\end{aligned}$$

Here α is a scale parameter, while β and θ are shape parameters. The corresponding pdf is given by

$$\begin{aligned}g(x;\alpha,\beta,\theta,p) &= (1-p)\theta\beta\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}\\ &\quad\times\frac{\left(1-e^{-(\alpha x)^{\beta}}\right)^{\theta-1}}{\left[1-p\left(1-e^{-(\alpha x)^{\beta}}\right)^{\theta}\right]^{2}}. \qquad (6)\end{aligned}$$

We shall refer to the distribution given by Equations (5) and (6) as the exponentiated Weibull geometric (EWG) distribution which was introduced by (MAHMOUDI; SHIRMAN, 2012). If a random variable X has the EWG distribution, then we write $X\sim EWG(\alpha,\beta,\theta,p)$. Replacing $G(x;\phi)$ in equation (1) by the cdf (5) yields the new cdf

$$\begin{aligned}&F_{_{BEWG}}(x,\alpha,\beta,\theta,p,a,b)=I_{\{G(x;\alpha,\beta,\theta,p)\}}(a,b)\\ &=\frac{1}{B(a,b)}\int_{0}^{G(x)}w^{(a-1)}(1-w)^{b-1}dw,x>0, \qquad (7)\end{aligned}$$

where $G(x)$ is given by equation (5), for $a>0,b>0$. A random variable X with the cdf (7) is said to have a Beta exponentiated Weibull geometric (BEWG) distribution and shall be denoted by $X \sim$ BEWG(φ) where $\varphi = (\alpha, \beta, \theta, p, a, b)$. The corresponding pdf $f_{_{BEWG}}(x, \varphi)$ of the new distribution takes the form

$$\begin{aligned} &(1-p)^b \theta\beta\alpha^\beta x^{\beta-1} e^{-(\alpha x)^\beta} \left(1 - e^{-(\alpha x)^\beta}\right)^{\theta b - 1} \\ &\qquad \times \frac{\left[1 - \left(1 - e^{-(\alpha x)^\beta}\right)^\theta\right]^{a-1}}{B(a,b)\left[1 - p\left(1 - e^{-(\alpha x)^\beta}\right)^\theta\right]^{a+b}}, \end{aligned} \tag{8}$$

where the parameters $a > 0$ and $b > 0$ are shape parameters, which characterize the skewness, kurtosis, and unimodality of the distribution.

The reliability (survival) function (RF) of the BEWG distribution is denoted by $R_{BEWG}(x)$ and it is defined as

$$R_{_{BEWG}}(x, \varphi) = 1 - F_{_{BEWG}}(x, \varphi) = I_{\{1 - G(x; \alpha, \beta, \theta)\}}(a, b). \tag{9}$$

The BEWG hazard rate function (HF), $h_{_{BEWG}}(x, \varphi)$, is given by

$$\begin{aligned} &\frac{(1-p)^b \theta\beta\alpha^\beta x^{\beta-1} e^{-(\alpha x)^\beta} \left(1 - e^{-(\alpha x)^\beta}\right)^{\theta b - 1}}{B(a,b)\left[1 - p\left(1 - e^{-(\alpha x)^\beta}\right)^\theta\right]^{a+b}} \\ &\qquad \times \frac{\left[1 - \left(1 - e^{-(\alpha x)^\beta}\right)^\theta\right]^{a-1}}{I_{\{1 - G(x; \alpha, \beta, \theta)\}}(a, b)}. \end{aligned} \tag{10}$$

2.1 Special cases of the BEWG distribution

The beta exponentiated Weibull geometric distribution is very flexible model that approaches to different distributions when its parameters are changed. The BEWG distribution contains as special- models the following well known distributions. If X is a random variable with pdf (8), then we have the following special cases.

1. For $\theta = 1$ we get the beta Weibull geometric (BWG) distribution which introduced by (BIDRAM; BEHBOODIAN; TOWHIDI, 2013). In addition , when $p \downarrow 0$, the beta Weibull proposed by (CORDEIRO; SILVA; ORTEGA, 2011) is obtained.

2. If $\beta = 1$, the BEWG distribution gives a new lifetime distribution called beta generalized exponential geometric (BGEG) distribution which proposed by (see also (CORDEIRO; SILVA; ORTEGA, 2011)). In addition the generalized exponential geometric (GEG) distribution is obtained when $a = b = 1$, and if $p \downarrow 0$ an generalized exponential distribution with parameter $\alpha > 0$ and $\theta > 0$ is achieved.

3. If $\theta = \beta = 1$, the BEWG distribution gives a new lifetime distribution called beta exponential geometric (BEG) distribution which proposed by (BIDRAM; BEHBOODIAN; TOWHIDI, 2013). In addition the exponential geometric (EG) distribution is obtained when $a = b = 1$, and if $p \downarrow 0$ an beta exponential distribution with parameter $\alpha > 0$ is obtained.

4. Beta generalized Rayleigh geometric (BGRG) arises as a special case of BEWG by taking $\beta = 2$ (see (CORDEIRO et al., 2013)). In addition , For $\theta = 1$ and $\beta = 2$ we get the beta Rayleigh geometric (BRG) distribution, when $p \downarrow 0$, the beta Rayleigh is obtained.

5. For $a = b = 1$ we get the exponentiated Weibull geometric (EWG) distribution which introduced by (MAHMOUDI; SHIRMAN, 2012). In addition , when $p \downarrow 0$, the exponentiated Weibull is obtained (see (NADARAJAH; CORDEIRO; ORTEGA, 2013)).

2.2 Expansion for the cumulative and density functions

In this subsection we present some representations of cdf, pdf of BEWG distribution. The mathematical relation given below will be useful in this subsection. Here and henceforth, let X be a random variable having the $BEWG(\varphi)$, we can obtain some alternative expressions for $F(x)$ and $f(x)$ of the BEWG distribution. The series representation given below will be useful in this subsection. If b is a positive real non- integer and $|z| < 1$ then

$$(1-z)^{b-1} = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(b)}{\Gamma(b-i)i!} z^i \tag{11}$$

One can re-express Equations (7) as

$$\begin{aligned} F_{BEWG}(x,\varphi) &= \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(b)}{\Gamma(b-i)i!(a+i)} \\ &\quad \times [G(x;\alpha,\beta,\theta)]^{a+i} \end{aligned} \tag{12}$$

$$\begin{aligned} &= \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(b)}{\Gamma(b-i)i!(a+i)} \\ &\quad \times F_{GBEWG}(x;\alpha,\beta,\theta,a+i). \end{aligned} \tag{13}$$

Also, if b is a positive real integer values, then the upper of this summation is finite and stops at $b-1$. Thus

$$\begin{aligned} F_{BEWG}(x,\varphi) &= \frac{[G(x;\alpha,\beta,\theta)]^a}{\Gamma(a)} \sum_{i=0}^{b-1} \frac{\Gamma(a+i)}{i!} \\ &\quad \times [G(x;\alpha,\beta,\theta)]^i . \end{aligned} \tag{14}$$

Using the series representation

$$(1-z)^{-k} = \sum_{i=0}^{\infty} \frac{\Gamma(k+i)}{\Gamma(k) i!} z^i, |z| < 1, k > 0, \tag{15}$$

the pdf (8) can be expressed as

$$\begin{aligned} f_{{}_{BEWG}}(x,\varphi) &= \frac{(1-p)^b}{\Gamma(b)} \sum_{i=0}^{\infty} \frac{\Gamma(b+i) p^{i}}{i!} \theta\beta\alpha^{\beta} x^{\beta-1} \\ & \times e^{-(\alpha x)^{\beta}} \Big(1 - e^{-(\alpha x)^{\beta}}\Big)^{\theta(b+i)-1} \\ & \times \left[1 - \Big(1 - e^{-(\alpha x)^{\beta}}\Big)^{\theta}\right]^{a-1} \\ &= \frac{(1-p)^b}{\Gamma(b)} \sum_{i=0}^{\infty} \frac{\Gamma(b+i) p^{i}}{i!} \\ & \times f_{{}_{BEW}}(x, \alpha, \beta, \theta, a, b+i), \end{aligned} \tag{16}$$

where $f_{{}_{BEW}}(x, \alpha, \beta, \theta, a, b+i)$ is the pdf of the beta exponentiated (generalized) Weibull distribution. As we see, the pdf of the BEWG distribution is an infinite mixture of BEW densities (see (CORDEIRO et al., 2013)). Hence, we can obtain some mathematical properties of the BEWG distribution from those properties of the BEW distribution.

3 Statistical properties

In this section we discuss the statistical properties of the BEWG distribution. Specifically, moments, moment generating function, mean deviation. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

Let the random variable Y follow the BEW $(\alpha, \beta, \theta, a, b)$ distribution. (CORDEIRO et al., 2013) showed that the r_{th} momen of Y (for both a and θ real non-integers) is

$$\mu_r^{'} = E(Y^r) = \Gamma(\frac{r}{\beta} + 1) \sum_{k=1}^{\infty} w_{+,k} \alpha_k^{-r} \tag{17}$$

where

$$w_{+,k} = \sum_{j=0}^{\infty} w_{j,k} = \sum_{j=0}^{\infty} \frac{(-1)^{j+k+1} \binom{b-1}{j} \binom{\theta(a+j)}{k}}{(a+j) B(a,b)}$$

and $\alpha_k = \alpha\ k^{\frac{1}{\beta}}$ is the scale parameter. For $a = b = \theta = 1$, Equation (17) yields precisely the r_{th} momen of the Weibull distribution. Hence, combining Equations (16) and (17), we obtain the r_{th} moment of the BEWG distribution

$$\begin{aligned}
\mu_r^{'} &= E(X^r) = \frac{(1-p)^b}{\Gamma(b)} \\
&\quad \times \sum_{i=0}^{\infty} \frac{\Gamma(b+i)p^{i}}{i!} \Gamma(\frac{r}{\beta}+1) \sum_{k=1}^{\infty} w_{+,k} \alpha_k^{-r} \\
&= \frac{(1-p)^b}{\Gamma(b)} \sum_{i=0}^{\infty} \frac{\Gamma(b+i)p^{i}}{i!} \Gamma(\frac{r}{\beta}+1) \\
&\quad \times \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{j+k+1} \binom{b+i-1}{j} \binom{\theta(a+j)}{k}}{(a+j)B(a,b+i)}. \qquad (18)
\end{aligned}$$

(CORDEIRO et al., 2013) also showed that the moment generating function of $Y \sim$ BEW$(\alpha, \beta, \theta, a, b)$ are

$$M_Y(t) = \frac{\theta \alpha^{\beta}}{B(a,b)} \sum_{k=0}^{\infty} \frac{f_k}{\xi_k^{\beta}}\ {}_1\Psi_0 \left[\begin{array}{c} (1, \beta^{-1}) \\ - \end{array} ; \frac{t}{\xi_k} \right], \qquad (19)$$

where

$${}_p\Psi_q \left[\begin{array}{c} (\mu_1, A_1), ..., (\mu_p, A_p) \\ (\lambda_1, B_1), ..., (\lambda_p, B_p) \end{array} ; x \right] = \sum_{n=0}^{\infty} \frac{\Pi_{j=1}^{p} \Gamma(\mu_j + A_j n) x^n}{\Pi_{j=1}^{q} \Gamma(\lambda_j + B_j n) n!}$$

is called Wright-generalized hypergeometric function. We combine the equations (16) and (19), and obtain the moment generating function of the BEWG distribution

$$\begin{aligned}
M_X(t) &= \frac{(1-p)^b}{\Gamma(b)} \sum_{i=0}^{\infty} \frac{\Gamma(b+i)p^{i}}{i!} \frac{\theta \alpha^{\beta}}{B(a,b)} \\
&\quad \times \sum_{k=0}^{\infty} \frac{f_k}{\xi_k^{\beta}}\ {}_1\Psi_0 \left[\begin{array}{c} (1, \beta^{-1}) \\ - \end{array} ; \frac{t}{\xi_k} \right], \qquad (20)
\end{aligned}$$

and the characteristic function $\phi(t) = E\left(e^{itX}\right)$ of X corresponding to equation (19) is given by

$$\begin{aligned}
\phi(t) &= \frac{(1-p)^b}{\Gamma(b)} \sum_{i=0}^{\infty} \frac{\Gamma(b+i)p^{i}}{i!} \frac{\theta \alpha^{\beta}}{B(a,b)} \\
&\quad \times \sum_{k=0}^{\infty} \frac{f_k}{\xi_k^{\beta}}\ {}_1\Psi_0 \left[\begin{array}{c} (1, \beta^{-1}) \\ - \end{array} ; \frac{it}{\xi_k} \right]
\end{aligned}$$

provided that $\beta > 1$.

3.1 Mean deviation

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. If X has the BEWG distribution , then we can derive the mean deviations about the mean $\mu = E(X)$ and the mean deviations about the median M are defined by

$$\delta_1(x) = \int_0^\infty \mid x - \mu \mid f(x)dx$$

and

$$\delta_2(x) = \int_0^\infty \mid x - M \mid f(x)dx$$

respectively. The measures $\delta_1(x)$ and $\delta_2(x)$ can be calculated using the relationships

$$\begin{aligned}\delta_1(x) &= \int_0^\infty \mid x - \mu \mid f(x)dx \\ &= \int_0^\mu (\mu - x)f(x)dx + \int_\mu^\infty (x-\mu)f(x)dx \\ &= \mu F(\mu) - \int_0^\mu xf(x)dx - \mu\left[1 - F(\mu)\right] \\ &\quad \times + \int_\mu^\infty xf(x)dx \\ &= 2\left[\mu F(\mu) - J(\mu)\right]. \qquad (21)\end{aligned}$$

And

$$\delta_2(x) = \int_0^\infty \mid x - M \mid f(x)dx = \mu - 2J(M). \qquad (22)$$

Following (CORDEIRO et al., 2013) and considering that

$$J(a) = \sum_{k=1}^{\infty} \frac{w_{+,k}}{\alpha_k}\gamma(\beta^{-1}+1,(a\alpha_k)^\beta,$$

where $\gamma(\alpha, b)$ is the lower incomplete gamma function given by $\gamma(\alpha, b) = \int_0^b t^{\alpha-1}e^{-t}dt$, and the mean μ is obtained from equation (18), $\delta_1(x)$ and $\delta_2(x)$ are given by

$$\begin{aligned}\delta_1(x) &= \frac{2(1-p)^b}{\Gamma(b)} \\ &\times \sum_{i=0}^{\infty} \frac{\Gamma(b+i)p^{i'}}{i!}\left[\mu F(\mu) - \sum_{k=1}^{\infty}\frac{w_{+,k}}{\alpha_k}\gamma(\beta^{-1}+1,(a\alpha_k)^\beta\right].\end{aligned}$$

and

$$\delta_2(x) = \frac{(1-p)^b}{\Gamma(b)} \times \sum_{i=0}^{\infty} \frac{\Gamma(b+i)p^{i‘}}{i!} \left[\mu - 2\sum_{k=1}^{\infty} \frac{w_{+,k}}{\alpha_k} \gamma(\beta^{-1}+1, (M\alpha_k)^{\beta}) \right],$$

respectively.

3.2 Bonferroni and Lorenz curves

The Bonferroni and Lorenz curves (BONFERRONI, 1930) and the Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined by

$$\begin{aligned} B(\pi) &= \frac{J(q)}{\pi\mu} \qquad (23) \\ &= \frac{(1-p)^b}{\Gamma(b)} \sum_{i=0}^{\infty} \frac{\Gamma(b+i)p^{i‘}}{i!} \frac{1}{\pi\mu} \\ &\quad \times \sum_{k=1}^{\infty} \frac{w_{+,k}}{\alpha_k} \gamma(\beta^{-1}+1, (q\alpha_k)^{\beta}) \end{aligned}$$

and

$$\begin{aligned} L(p) &= \frac{J(q)}{\mu} \qquad (24) \\ &= \frac{(1-p)^b}{\Gamma(b)} \sum_{i=0}^{\infty} \frac{\Gamma(b+i)p^{i‘}}{i!} \\ &\quad \times \sum_{k=1}^{\infty} \frac{w_{+,k}}{\alpha_k} \gamma(\beta^{-1}+1, (q\alpha_k)^{\beta}), \end{aligned}$$

respectively.

4 Entropy

The entropy of a random variable X with density $f(x)$ is a measure of variation of the uncertainty. A large value of entropy indicates a great uncertainty in the data. The Rényi entropy is defined as

$$I_R(\rho) = \frac{1}{1-\rho} \log \left\{ \int f(x)^{\rho} dx \right\},$$

where $\rho > 0$ and $\rho \neq 1$. (CORDEIRO et al., 2013) showed that the Rényi and Shannon entropy of beta exponentiated Weibull are given by

$$I_R(\rho)_{BEW} = \frac{\rho}{1-\rho}\log\left[\frac{\theta}{\log B(a,b)}\right] - \log(\alpha\beta)$$
$$+\frac{1}{1-\rho}\log\left\{\Gamma(\frac{\rho(\beta-1)+1}{\beta}\right.$$
$$\times \sum_{j,k=0}^{\infty}\frac{(-1)^{j+k}}{(k+\rho)^{\rho-(\rho-1)/\beta}}\binom{\rho(b-1)}{j}\binom{\rho(\theta a-1)+j\theta}{k}, \quad (25)$$

and

$$E\left[-\log f(x)\right]_{BEW} = -\log\left(\theta\beta\alpha^{\beta}\right) + \log(B(a+b)) \quad (26)$$
$$+(\beta-1)\left[\frac{\gamma}{\beta} + \frac{1}{\beta}\sum_{k=1}^{\infty} w_{+,k}\log(k) + \log(\alpha)\right]$$
$$+\sum_{k=1}^{\infty} k^{-1} w_{+,k} - \frac{\theta a-1}{\theta}\left[\psi(a) - \psi(a+b)\right]$$
$$-(b-1)\left[\psi(b) - \psi(a+b)\right],$$

respectively, where $\psi(z) = \frac{d\log(\Gamma(Z))}{dZ}$ is the digamma function and γ is Euler´s constant. We combining equations (16) , (25) and (26) we get the Rényi and Shannon entropy of beta exponentiated Weibull geometric as follows

$$I_R(\rho)_{BEWG} = \frac{(1-p)^b}{\Gamma(b)}\sum_{i=0}^{\infty}\frac{\Gamma(b+i)p^{i}}{i!} I_R(\rho)_{BEW}$$

and

$$E\left[-\log f(x)\right]_{BEWG} = \frac{(1-p)^b}{\Gamma(b)}\sum_{i=0}^{\infty}\frac{\Gamma(b+i)p^{i}}{i!}$$
$$\times E\left[-\log f(x)\right]_{BEW},$$

respectively.

5 Estimation and inference

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the BEWG distribution. Let $X_1, X_2, ..., X_n$ be a random sample of size n from BEWG(φ), where $\varphi = (\alpha, \beta, \theta, p, a, b)$. Let $\varphi = (\alpha, \beta, \theta, p, a, b)^T$ be the parameter vector. The log likelihood function for the vector of parameters $\varphi = (\alpha, \beta, \theta, p, a, b)$ can be written as

$$\begin{aligned}
\log\ L &= nb\log(1-p) + n\log(\theta) + n\log(\beta) + n\beta\log(\alpha) \\
&\quad - \sum_{i=1}^{n}(\alpha x_i)^{\beta} + (\beta-1)\sum_{i=1}^{n}\log x_i - n\log\left[B(a,b)\right] \\
&\quad + (\theta b - 1)\sum_{i=1}^{n}\log\left(1 - e^{-(\alpha x_i)^{\beta}}\right) \\
&\quad - (a+b)\sum_{i=1}^{n}\log\left[1 - p\left(1 - e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\right]. \qquad (27)
\end{aligned}$$

The associated score function is given by

$$U_n(\varphi) = \left[\frac{\partial L}{\partial \alpha}, \frac{\partial L}{\partial \beta}, \frac{\partial L}{\partial \theta}, \frac{\partial L}{\partial p}, \frac{\partial L}{\partial a}, \frac{\partial L}{\partial b}\right]^T.$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (27). The components of the score vector are given by

$$\begin{aligned}
\frac{\partial \log\ L}{\partial \alpha} &= \frac{n\beta}{\alpha} - \beta x_i \sum_{i=1}^{n}(\alpha x_i)^{\beta-1} \\
&\quad + \beta(\theta b - 1)\sum_{i=1}^{n}\frac{x_i(\alpha x_i)^{\beta-1}e^{-(\alpha x_i)^{\beta}}}{\left(1 - e^{-(\alpha x_i)^{\beta}}\right)} \\
&\quad - (a+b)\sum_{i=1}^{n} p\theta\beta x_i(\alpha x_i)^{\beta-1}e^{-(\alpha x_i)^{\beta}} \\
&\quad \times \frac{\left(1 - e^{-(\alpha x_i)^{\beta}}\right)^{\theta-1}}{\left[1 - p\left(1 - e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\right]}, \qquad (28)
\end{aligned}$$

$$\begin{aligned}\frac{\partial \log\ L}{\partial\beta} &= \frac{n}{\beta} + n\log(\alpha) - \sum_{i=1}^{n}(\alpha x_i)^{\beta}\log(\alpha x_i) + \sum_{i=1}^{n}\log x_i \\ &+(a+b)\sum_{i=1}^{n} -p\theta(\alpha x_i)^{\beta}\log(\alpha x_i)e^{-(\alpha x_i)^{\beta}} \\ &\times\frac{\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta-1}}{\left[1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\right]} \\ &+(\theta b-1)\sum_{i=1}^{n}\frac{(\alpha x_i)^{\beta}\log(\alpha x_i)e^{-(\alpha x_i)^{\beta}}}{\left(1-e^{-(\alpha x_i)^{\beta}}\right)},\end{aligned} \tag{29}$$

$$\begin{aligned}\frac{\partial \log\ L}{\partial\theta} &= \frac{n}{\theta} + b\sum_{i=1}^{n}\log\left(1-e^{-(\alpha x_i)^{\beta}}\right) \\ &-(a+b)\sum_{i=1}^{n} p\log\left(1-e^{-(\alpha x_i)^{\beta}}\right) \\ &\times\frac{\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}}{\left[1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\right]},\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{\partial \log\ L}{\partial p} &= \frac{-nb}{(1-p)} \\ &+(a+b)\sum_{i=1}^{n}\frac{\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}}{\left[1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\right]},\end{aligned} \tag{31}$$

$$\begin{aligned}\frac{\partial \log\ L}{\partial a} &= n\psi(a+b) - n\psi(a) \\ &-\sum_{i=1}^{n}\log\left[1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\right],\end{aligned} \tag{32}$$

and

$$\frac{\partial \log\ L}{\partial b} = n\log(1-p) + n\psi(a+b) + \theta\sum_{i=1}^{n}\log\left(1-e^{-(\alpha x_i)^{\beta}}\right) - n\psi(b) - \sum_{i=1}^{n}\log\left[1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\right]. \tag{33}$$

The maximum likelihood estimation (MLE) of φ, say $\widehat{\varphi}$, is obtained by solving the nonlinear system $U_n(\varphi) = 0$. These equations cannot be solved analytically, and statistical software can be used to solve them numerically via iterative methods. We can use iterative techniques such as a Newton–Raphson type algorithm to obtain the estimate $\widehat{\varphi}$. The Broyden–Fletcher–Goldfarb–Shanno method with analytical derivatives has been used for maximizing the log-likelihood function $L(\varphi)$. For interval estimation and hypothesis tests on the distribution parameters, the information matrix is required. The 6×6 observed information matrix is given by

$$I_n(\varphi) = -\begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\theta} & I_{\alpha p} & I_{\alpha a} & I_{\alpha b} \\ I_{\beta\alpha} & I_{\beta\beta} & I_{\beta\theta} & I_{\beta p} & I_{\beta a} & I_{\beta b} \\ I_{\theta\alpha} & I_{\theta\beta} & I_{\theta\theta} & I_{\theta p} & I_{\theta a} & I_{\theta b} \\ I_{p\alpha} & I_{p\beta} & I_{p\theta} & I_{pp} & I_{pa} & I_{pb} \\ I_{a\alpha} & I_{a\beta} & I_{a\theta} & I_{ap} & I_{aa} & I_{ab} \\ I_{b\alpha} & I_{b\beta} & I_{b\theta} & I_{bp} & I_{ba} & I_{bb} \end{bmatrix},$$

whose elements of Hessian matrix are given in Appendix A. Applying the usual large sample approximation, $\widehat{\varphi}$ can be treated as being approximately $N_6(\varphi, J_n(\varphi)^{-1})$, where $J_n(\varphi) = E\left[I_n(\varphi)\right]$. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\widehat{\varphi} - \varphi)$ is $N_6(0, J(\varphi)^{-1})$, where $J(\varphi) = \lim_{n\to\infty} n^{-1}I_n(\varphi)$ is the unit information matrix. This asymptotic behavior remains valid if $J(\varphi)$ is replaced by the average sample information matrix evaluated at $\widehat{\varphi}$, say $n^{-1}I_n(\widehat{\varphi})$. The estimated asymptotic multivariate normal $N_6(\varphi, I_n(\widehat{\varphi})^{-1})$ distribution of $\widehat{\varphi}$ can be used to construct approximate confidence intervals for the parameters. An $100(1-\xi)$ asymptotic confidence interval for each parameter φ_r is given by

$$ACI_r = \left(\widehat{\varphi}_r - z_{\frac{\xi}{2}}\sqrt{\widehat{I_{rr}}}, \widehat{\varphi}_r + z_{\frac{\xi}{2}}\sqrt{\widehat{I_{rr}}}\right),$$

where $\widehat{I_{rr}}$ is the (r, r) diagonal element of $I_n(\widehat{\varphi})^{-1}$ for $r = 1, 2, 3, 4, 5, 6$ and $z_{\xi/2}$ is the quantile $1 - (\xi/2)$ of the standard normal distribution.

6 Data experiments

Cervical intraepithelial neoplasia (CIN), also known as cervical dysplasia and cervical interstitial neoplasia, is the potentially premalignant transformation and abnormal growth (dysplasia) of squamous cells on the surface of the cervix. Most cases of CIN remain stable, or are eliminated by the host's immune system without intervention. However a small percentage of cases progress to become cervical cancer, usually cervical squamous cell carcinoma (SCC), if left untreated. The estimated annual incidence in the United States of CIN among women who undergo cervical cancer screening is 4 percent for CIN 1 and 5 percent for CIN 2,3, (AGORASTOS et al., 2005). It is estimated that about 500,000 new cases are reported every year, with approximately 230,000 deaths worldwide. In Brazil, the crude incidence rates per 100,000 women, estimated for the year 2012, were 17 for the country and 14 for the Rio Grande do Norte State. The incidence of the disease starts from the age of 20 and the risk gradually increases with age, reaching its peak generally at age 50 to 60, (LIMA et al., 2013).

The presented methodology in this paper is applied to a real study involving patients treated in the Woman Clinic sited in Maringá city, Paraná State, Brazil. The data consist of the time until the cure of CIN (in months) in 363 women followed up from years 2,000 to 2,006. For all patients was observed the maximum time that the initial lesions took to be considered totally cure (after specific diagnostics). We consider a new model that is a generalization to describe the lifetime.

Firstly, a brief descriptive analysis was made. The minimum observed time was 1 month and the maximum observed one was 47 months, approximately four years. The mean time for the cure is 6.4 months with standard deviation 5.099. The left panel of the Figure 1, top left panel, shows the TTT plot (for example see (BARLOW; CAMPO, 1975)), in order to verify the possible shape for the hazard function. If the TTT plot is concave and then convex, it indicates unimodal hazard, which can be accommodated by a BEWG distribution.

Then, the BEWG model was fitted and the estimates of the parameters and errors can be seen in Table 1. In order to illustrate some special cases, presented in Section 2.1, the models BWG and BW were fitted too and the estimates are presented in Table 1.

In order to select the most appropriate model to describe the time to cure, two different criterias were used: AIC and BIC. The AIC for BEWG, BWG and BW models were, respectively, 1940.0, 1938.0 and 1936.5; and the BIC were, 1963.4, 1957.5 and 1952,1, respectively. Also, by considering the likelihood-ratio test, there is no significant difference between the presented models. Thus, as the main idea of this study is to show the applicability of the BEWG distribution and their nested models, in this paper we will use the main model, BEWG, in the description of the time to cure patients.

Figure 1, bottom left panel, shows the empirical curve obtained by Kaplan-Meier method versus the estimated curves by the model BEWG and the curves are very close indicating a good fit. Also, in right panel we can see the histogram and

Table 1 - MLEs for the parameters of the BEWG distribution and the confidence intervals

Model	Parameter	Estimate	Error	Confidence Intervals 95% Lower	Upper
BEWG	a	0.849	0.131	0.592	1.106
	b	4.851	0.048	4.757	4.946
	p	0.944	0.351	0.253	1.634
	θ	0.607	3.136	-5.560	6.773
	α	5.291	8.455	-11.336	21.919
	β	0.457	0.134	0.194	0.720
BWG	a	0.749	0.758	-0.741	2.238
	b	4.401	0.906	2.619	6.184
	p	0.895	0.341	0.223	1.566
	α	3.984	3.279	-5.066	7.831
	β	0.498	0.431	-0.350	1.345
BW	a	1.151	0.079	0.995	1.307
	b	13.738	0.217	13.311	14.164
	α	1.381	0.126	1.133	1.629
	β	0.534	0.021	0.492	0.576

the pdf estimated by the model and the estimated hazard curve.

Moreover, as a goodness-of-fit procedure, we performed a residuals analysis for the BEWG model by using the Cox-Snell residuals (COX; SNELL, 1968). The Cox-Snell residuals are defined as $\hat{e}_i = \hat{\Lambda}(t_i \mid x_i)$, where $\hat{\Lambda}(.)$ is the cumulative hazard function of the adjusted model. If we consider the BEWG model, the Cox-Snell residual is given by

$$\hat{e}_{iBEWG} = -\ln\left[I_{\{1-G(x;\alpha,\beta,\theta)\}}(a,b)\right]. \tag{34}$$

Then, the deviance residual and t versus $-\log(S(t))$ are showed in Figures 2 left and right panels, respectively. Note that the Figure 2, right panel, shows a linear behavior as expected for a good fit and no outliers were observed in Figure 2 left panel.

Finally, we describe the results of a simulation study designed to asses the frequentist properties of the model BEWG in order to validate the results showed above.

Consider the sample size 363 of patients that were treated in Woman Clinic. Then, a bootstrap study was made considering 6 different sample sizes: 50, 80, 100, 150, 200 and 500. For each sample size we obtained $1,000$ resampling which the BEGW model was fitted. The relative difference between the estimated parameters by using the original sample and the bootstrap samples, BIAS, mean square error (MSE) and coverage probability are presented in Table 2.

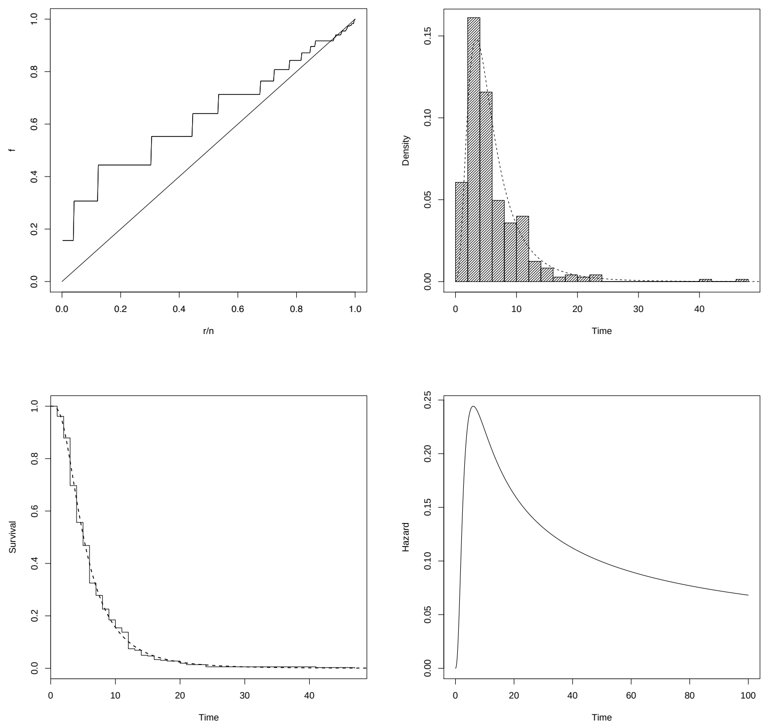

Figure 1 - TTTPlot; Histogram with the BEGW pdf estimated; Empirical and estimated by BEGW survival curves and; Hazard curve estimated by the model BEGW.

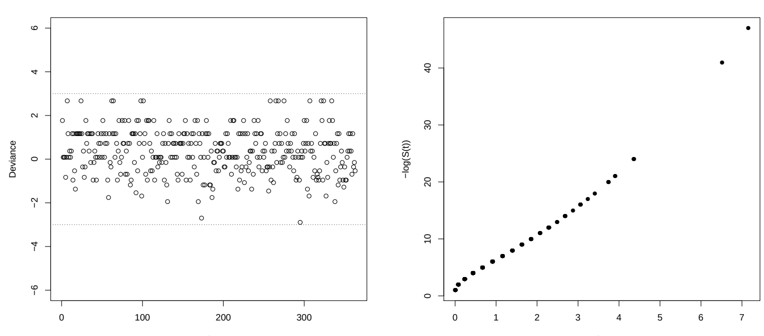

Figure 2 - Left panel: Deviance residuals; Right panel: t $\times -\log(S(t))$.

Table 2 - Simulation bootstrap

Sample Size	Parameter	Relative Difference	BIAS	MSE	Coverage Probability
50	a	0.805	3.498	18.923	0.837
	b	0.202	0.981	6.727	0.923
	p	0.142	0.134	0.306	0.873
	θ	0.293	0.178	0.224	0.811
	α	0.201	1.065	4.818	0.957
	β	0.448	0.205	0.741	0.986
80	a	0.765	2.762	13.501	0.857
	b	0.191	0.929	6.331	0.923
	p	0.126	0.119	0.287	0.875
	θ	0.261	0.158	0.217	0.858
	α	0.144	0.762	4.163	0.962
	β	0.291	0.133	0.403	0.991
100	a	0.741	2.434	11.267	0.868
	b	0.252	1.224	7.039	0.918
	p	0.117	0.111	0.271	0.887
	θ	0.286	0.173	0.209	0.772
	α	0.091	0.482	3.908	0.953
	β	0.190	0.087	0.281	0.983
150	a	0.664	1.681	7.262	0.898
	b	0.189	0.919	5.838	0.931
	p	0.086	0.081	0.232	0.906
	θ	0.251	0.152	0.202	0.807
	α	0.020	0.108	3.666	0.945
	β	0.141	0.065	0.211	0.934
200	a	0.644	1.539	6.538	0.905
	b	0.209	1.016	6.029	0.924
	p	0.086	0.081	0.227	0.908
	θ	0.239	0.145	0.190	0.803
	α	0.030	0.160	3.540	0.948
	β	0.109	0.050	0.174	0.919
500	a	0.280	0.330	1.807	0.956
	b	0.058	0.280	3.540	0.940
	p	0.039	0.037	0.132	0.951
	θ	0.183	0.111	0.159	0.833
	α	0.056	0.295	3.086	0.953
	β	0.097	0.044	0.145	0.930

Conclusion

In this paper we have introduced a new distribution called beta exponentiated Weibull-geometric (BEWG) distribution. The new distribution is generated from the logit of a beta random variable and includes the exponentiated Weibull geometric distribution of (MAHMOUDI; SHIRMAN, 2012). Some of its structural properties were presented such as moment generating function, mean deviation, entropy and Lorenz curve. Estimation of the models parameters were performed by maximum likelihood method. The usefulness of the model was presented in an real application on cervical intraepithelial neoplasia dataset.

A simulation study is performed by using the bootstrap approach, from which we learned that the maximum likelihood estimate biases and standard errors decrease with the increasing of the sample size, and that the coverage probabilities of a 95% two sided confidence intervals for the model parameters become closer to the nominal ones as the sample size increases.

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LOUZADA, F.; ELBATAL, I.; GRANZOTTO, D. C. T. A Distribuição Geométrica Exponencial de Weibull Beta: Modelagem, propriedades estruturais, estimação e uma aplicação a um conjunto de dados de neoplasia intraepitelial cervical. *Rev. Bras. Biom.,* Lavras, v.36, n.4, p.942-967, 2018.

- *RESUMO: Uma nova distribuição é proposta, a chamada distribuição Beta Exponenciada Weibull Geométrica (BEWG). A nova distribuição é gerada a partir do logito de uma variável aleatória Beta e inclui a distribuição Geométrica Weibull exponenciada como caso particular. Várias propriedades estruturais, incluindo expressões explícitas para o momentos, função geradora de momento, desvios médios da nova distribuição são derivadas. A estimação dos parâmetros do modelo é realizada por máxima verossimilhança método. A utilidade do modelo foi mostrada usando um conjunto de dados real. A fim de validar os resultados uma simulaçã bootstrap é apresentada neste documento.*

- *PALAVRAS-CHAVE: Estimação de máxima verossimilhança; função geradora de momento; função de confiabilidade.*

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A Hessian Matrix

The Hessian matrix is given by

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ & & A_{33} & A_{34} & A_{35} & A_{36} \\ & & & A_{44} & A_{45} & A_{46} \\ & & & & A_{55} & A_{56} \\ & & & & & A_{66} \end{bmatrix}$$

where

$$\begin{aligned} A_{11} \quad = \quad & \frac{\partial^2 \log L}{\partial \alpha^2} = -\frac{n\beta}{\alpha^2} - \sum_{i=1}^{n} (\beta - 1) \frac{\beta(\alpha x_i)^\beta}{\alpha^2} \\ & +\theta(b-1) \sum_{i=1}^{n} \frac{(\alpha x_i)^\beta \beta e^{-(\alpha x_i)^\beta}}{\alpha^2 [1 - e^{-(\alpha x_i)^\beta}]} \left[\beta - 1 - \beta(\alpha x_i)^\beta \left(1 - \frac{e^{-(\alpha x_i)^\beta}}{1 - e^{-(\alpha x_i)^\beta}} \right) \right] \\ & -(a+b) \sum_{i=1}^{n} \frac{p(1 - e^{-(\alpha x_i)^\beta})^\theta \theta (\alpha x_i)^\beta \beta e^{-(\alpha x_i)^\beta}}{\alpha^2 \left[1 - e^{-(\alpha x_i)^\beta}\right] \left(1 - p\left(1 - e^{-(\alpha x_i)^\beta}\right)^\theta\right)} \left[1 - \beta + \beta(\alpha x_i)^\beta \right. \\ & \left. + \left(e^{-(\alpha x_i)^\beta} - \theta \right) \frac{\beta(\alpha x_i)^\beta}{1 - e^{-(\alpha x_i)^\beta}} + \frac{p\left(1 - e^{-(\alpha x_i)^\beta}\right) \theta \beta (\alpha x_i)^\beta e^{-(\alpha x_i)^\beta}}{\left(1 - e^{-(\alpha x_i)^\beta}\right)\left(1 - p\left(1 - e^{-(\alpha x_i)^\beta}\right)\right)} \right] \end{aligned}$$

$$\begin{aligned} A_{12} \quad = \quad & \frac{\partial^2 \log L}{\partial \alpha \partial \beta} = \frac{n}{\alpha} - \sum_{i=1}^{n} \frac{(\alpha x_i)^\beta}{\alpha} \left(1 + \beta \log(\alpha x_i)\right) \\ & +(\theta b - 1) \sum_{i=1}^{n} \left(\frac{(\alpha x_i)^\beta e^{-(\alpha x_i)^\beta}}{\alpha \left(1 - e^{-(\alpha x_i)^\beta}\right)} \right) \times \\ & \times \left[\beta \log(\alpha x_i) + 1 - (\alpha x_i)^\beta \beta \log(\alpha x_i) \left(1 + \frac{e^{-(\alpha x_i)^\beta}}{1 - e^{-(\alpha x_i)^\beta}} \right) \right] \\ & -(a+b) \sum_{i=1}^{n} \frac{p\left(1 - e^{-(\alpha x_i)^\beta}\right)^\theta \theta (\alpha x_i)^\beta e^{-(\alpha x_i)^\beta}}{\alpha \left(1 - e^{-(\alpha x_i)^\beta}\right)\left(1 - p(1 - e^{-(\alpha x_i)^\beta})^\theta\right)} \times \\ & \times \left\{ -1 - \beta \log(\alpha x_i) \left[1 - (\alpha x_i)^\beta \left[1 + \frac{e^{-(\alpha x_i)^\beta}}{1 - e^{-(\alpha x_i)^\beta}} \left(1 - \theta - \frac{p\theta(1 - e^{-(\alpha x_i)^\beta})^\theta}{1 - p\left(1 - e^{-(\alpha x_i)^\beta}\right)} \right) \right] \right] \right\} \end{aligned}$$

$$A_{13} = \frac{\partial^2 \log L}{\partial\alpha\partial p} = (a+b)\sum_{i=1}^{n}\frac{\theta(\alpha x_i)^{\beta}\beta e^{-(\alpha x_i)^{\beta}}\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}}{\alpha\left(1-e^{-(\alpha x_i)^{\beta}}\right)\left(1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\right)}\times$$

$$\times\left(1+\frac{p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}}{\left(1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\right)}\right)$$

$$A_{14} = \frac{\partial^2 \log L}{\partial\alpha\partial\theta} = \sum_{i=1}^{n}\frac{(\alpha x_i)^{\beta}\beta e^{-(\alpha x_i)^{\beta}}}{\alpha\left(1-e^{-(\alpha x_i)^{\beta}}\right)} + (a+b)\sum_{i=1}^{n}\frac{p(\alpha x_i)^{\beta}\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\beta e^{-(\alpha x_i)^{\beta}}}{\alpha\left(1-e^{-(\alpha x_i)^{\beta}}\right)\left(1-p(1-e^{-(\alpha x_i)^{\beta}})^{\theta}\right)}\times$$

$$\times\left[1+\theta\log(1-e^{-(\alpha x_i)^{\beta}})\left[1+\frac{p(1-e^{(\alpha x_i)^{\beta}})^{\theta}}{1-p(1-e^{(\alpha x_i)^{\beta}})^{\theta}}\right]\right]$$

$$A_{15} = \frac{\partial^2 \log L}{\partial\alpha\partial a} = \sum_{i=1}^{n}\frac{p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\theta(\alpha x_i)^{\beta}\beta e^{-(\alpha x_i)^{\beta}}}{\alpha\left(1-e^{-(\alpha x_i)^{\beta}}\right)\left(1-p\left(1-e^{(\alpha x_i)^{\beta}}\right)^{\theta}\right)}$$

$$A_{16} = \frac{\partial^2 \log L}{\partial\alpha\partial b} = \theta\sum_{i=1}^{n}\frac{(\alpha x_i)^{\beta}\beta e^{-(\alpha x_i)^{\beta}}}{\alpha\left(1-e^{-(\alpha x_i)^{\beta}}\right)} + \sum_{i=1}^{n}\frac{p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\theta(\alpha x_i)^{\beta}\beta e^{-(\alpha x_i)^{\beta}}}{\alpha\left(1-e^{-(\alpha x_i)^{\beta}}\right)\left(1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\right)}$$

$$A_{22} = \frac{\partial^2 \log L}{\partial\beta^2} = -\frac{n}{\beta^2} - \sum_{i=1}^{n}(\alpha x_i)^{\beta}\log(\alpha x_i)^2$$

$$+(\theta b-1)\sum_{i=1}^{n}\left[\frac{(\alpha x_i)^{\beta}e^{-(\alpha x_i)^{\beta}}\log(\alpha x_i)^2}{1-e^{-(\alpha x_i)^{\beta}}}\left[1-(\alpha x_i)^{\beta}\left(\frac{e^{-(\alpha x_i)^{\beta}}}{1-e^{-(\alpha x_i)^{\beta}}}+1\right)\right]\right]$$

$$+(a+b)\sum_{i=1}^{n}\frac{p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\theta(\alpha x_i)^{\beta}\log(\alpha x_i)^2 e^{-(\alpha x_i)^{\beta}}}{\left(1-e^{-(\alpha x_i)^{\beta}}\right)\left(1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\right)}\times$$

$$\times\left[1-(\alpha x_i)^{\beta}\left[1+\frac{e^{-(\alpha x_i)^{\beta}}}{1-e^{-(\alpha x_i)^{\beta}}}\left(1-\theta-\frac{p\theta(1-e^{-(\alpha x_i)^{\beta}})^{\theta}}{1-p(1-e^{-(\alpha x_i)^{\beta}})^{\theta}}\right)\right]\right]$$

$$A_{23} = \frac{\partial^2 \log L}{\partial\beta\partial p} = (a+b)\sum_{i=1}^{n}\frac{\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\theta(\alpha x_i)^{\beta}\log(\alpha x_i)e^{-(\alpha x_i)^{\beta}}}{\left(1-e^{-(\alpha x_i)^{\beta}}\right)\left(1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\right)}\left[1+\frac{p\left(1-e^{-(\alpha x_i)^{\beta}}\right)}{1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}}\right]$$

$$
\begin{aligned}
A_{24} &= \frac{\partial^2 \log L}{\partial\beta\partial\theta} = b\sum_{i=1}^{n}\frac{(\alpha x_i)^{\beta}\log(\alpha x_i)e^{-(\alpha x_i)^{\beta}}}{1-e^{-(\alpha x_i)^{\beta}}} \\
&+(a+b)\sum_{i=1}^{n}\frac{p\left(1-e^{-(\alpha x_i)^{\beta}}\right)(\alpha x_i)^{\beta}\log(\alpha x_i)e^{-(\alpha x_i)^{\beta}}}{\left(1-e^{-(\alpha x_i)^{\beta}}\right)\left(1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\right)}\times \\
&\times\left[1+\theta\log(1-e^{-(\alpha x_i)^{\beta}})\left(1+\frac{p(1-e^{-(\alpha x_i)^{\beta}})^{\theta}}{1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}}\right)\right]
\end{aligned}
$$

$$
A_{25} = \frac{\partial^2 \log L}{\partial\beta\partial a} = \sum_{i=1}^{n}\frac{p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\theta(\alpha x_i)^{\beta}\log(\alpha x_i)e^{-(\alpha x_i)^{\beta}}}{\left(1-e^{-(\alpha x_i)^{\beta}}\right)\left(1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\right)}
$$

$$
A_{26} = \frac{\partial^2 \log L}{\partial\beta\partial b} = \theta\sum_{i=1}^{n}\frac{(\alpha x_i)^{\beta}\log(\alpha x_i)e^{-(\alpha x_i)^{\beta}}}{1-e^{-(\alpha x_i)^{\beta}}}+\sum_{i=1}^{n}\frac{p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\theta(\alpha x_i)^{\beta}\log(\alpha x_i)e^{-(\alpha x_i)^{\beta}}}{\left(1-e^{-(\alpha x_i)^{\beta}}\right)\left(1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\right)}
$$

$$
A_{33} = \frac{\partial^2 \log L}{\partial p^2} = -\frac{nb}{(1-p)^2}+(a+b)\sum_{i=1}^{n}\frac{\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{2\theta}}{\left[1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\right]^2}
$$

$$
A_{34} = \frac{\partial^2 \log L}{\partial p\partial\theta} = (a+b)\sum_{i=1}^{n}\frac{\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\log\left(1-e^{-(\alpha x_i)^{\beta}}\right)}{1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}}\left[1+\frac{p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}}{1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}}\right]
$$

$$
A_{35} = \frac{\partial^2 \log L}{\partial p\partial a} = \sum_{i=1}^{n}\left[\frac{\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}}{1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}}\right]
$$

$$
A_{36} = \frac{\partial^2 \log L}{\partial p\partial b} = -\frac{n}{1-p}+\sum_{i=1}^{n}\left[\frac{\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}}{1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}}\right]
$$

$$
A_{44} = \frac{\partial^2 \log L}{\partial\theta^2} = -\frac{n}{\theta^2}+(a+b)\sum_{i=1}^{n}\frac{p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}\log\left(1-e^{-(\alpha x_i)^{\beta}}\right)^2}{1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}}\left[1+\frac{\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}}{1-p\left(1-e^{-(\alpha x_i)^{\beta}}\right)^{\theta}}\right]
$$

$$A_{45} = \frac{\partial^2 \log L}{\partial\theta\partial a} = \sum_{i=1}^{n}\left[\frac{p\left(1-e^{-(\alpha x_i)^\beta}\right)^\theta \log\left(1-e^{-(\alpha x_i)^\beta}\right)}{1-p\left(1-e^{-(\alpha x_i)^\beta}\right)^\theta}\right]$$

$$A_{46} = \frac{\partial^2 \log L}{\partial\theta\partial b} = \sum_{i=1}^{n}\log\left(1-e^{-(\alpha x_i)^\beta}\right) + \sum_{i=1}^{n}\left[\frac{p\left(1-e^{-(\alpha x_i)^\beta}\right)^\theta \log\left(1-e^{-(\alpha x_i)^\beta}\right)}{1-p\left(1-e^{-(\alpha x_i)^\beta}\right)^\theta}\right]$$

$$A_{55} = \frac{\partial^2 \log L}{\partial a^2} = \Psi(1, a+b) - \Psi(1, a)$$

$$A_{56} = \frac{\partial^2 \log L}{\partial a\partial b} = \Psi(1, a+b)$$

$$A_{66} = \frac{\partial^2 \log L}{\partial b^2} = \Psi(1, a+b) - \Psi(1, b)$$

where $\Psi(\cdot)$ is the digamma function.