BETA CHI-SQUARE DISTRIBUTION PROPERTIES AND APPLICATIONS

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■ ABSTRACT: In this paper we introduce the distribution of three parameters, called Beta Chi-square distribution (BCHI), is presented and contains the Chi-square distribution as a sub-model. Its density function can be expressed as a linear combination of Chi-square density function. Some structural properties of this distribution, how moments, and hazard function are presented. Estimates of the model parameters are performed using the Maximum Likelihood method. We obtain the observed information matrix and discuss inference methods. In order to demonstrate the utility of the distribution, a real data set is analyzed.

■ KEYWORDS: Moments; Hazard function; Maximum Likelihood method; Observed information matrix.

1 Introduction

Lancaster (1966) observed that Bienaymé (1838) obtained the Chi-square distribution as the convergence in distribution of the random variable $\sum_{i=1}^{k}(N_i - np_i)^2/np_i$, where $N_1, \cdots, N_k$ has a multinomial joint distribution with the parameters $n, p_1, \cdots, p_k$. It is also known that $U_1, , U_2, \cdots, U_k$ are independent standard normal variables, so $\sum_{i=1}^{k} U_i^2$ has a Chi-square distribution with k degrees of freedom. Pearson's Chi-square distribution also appeared in (1900) as the approximate distribution for Chi-square statistics used for various tests on contingency tables (of course the distribution exact of the statistic is discrete). The use of the Chi-square distribution to approximate the distribution in a

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quadratic form (particularly the positive defined one) in multinormal variables is well established and generalized. Order moments and statistics of the Chi-square distribution appear in many areas of probability and statistics. Some examples are: approximations of convolutions of random variables, multivariate statistical tests, Pearson test statistics χ^2, quality of statistical adjustments, classification and selection procedures, tests of homogeneity of variances and transformations. For more details Tiku (1965), Jensen (1973), Hall (1983), Holtzman and Good (1986), Ko e Yum (1991), Wang (1994), and Mathai Pederzoli (1996), Fujikoshi (1997), Fujisawa (1997), Fujikoshi (2000) and Garcia-Perez and Nunez-Anton (2004).

The classes of beta-generalized distributions have received a considerable attention in recent years, particularly after the recent work by Eugene *et al.* (2002) and Jones (2004). Eugene *et al.* (2002) created the normal beta distribution (BN), introducing the cumulative distribution function of the normal $G(x)$ in (3), and derived some of its first moments. More general expressions of moment for BN were derived by Gupta and Nadarajah (2004). Nadarajah and Kotz (2004) presented the beta Gumbel distribution (BG) , obtained closed-form expressions for the moments,asymptotic distribution of extreme order statistics, and discussed the estimation procedure by maximum likelihood. Nadarajah and Gupta (2004) presented the beta Fréchet (BF) distribution for $G(x)$ being the Fréchet distribution.

Later, Nadarajah and Kotz (2005) worked with the exponential beta distribution (BE) and obtained the moment generating function, the first four cumulants, the asymptotic distribution of extreme order statistics and discussed the estimation procedure by maximum likelihood. The log F (or beta logistic) distribution was presented by Brown *et al.* (2002), existing for more than 20 years. Kong, *et al.* (2007) proposed a generalized distribution class called beta-gamma and examined some of its properties. Cordeiro and Nadarajah (2011) obtained closed forms of the expressions of moments of a class of generalized distributions and Cordeiro, *et al.* (2012) found the generalized beta gamma (BGG). Recently Elbatal, *et al.* (2019) introduced a new five parameter continuous probability distribution called the modified Gompertz beta distribution.

Chi-square distribution is a special case of the Gamma distribution then the Beta Chi-square distribution (BCHI), proposed here, is a special case of the Beta-gamma distribution. In this article, we present the BCHI distribution, analyze some of its properties, the shape of the pdf, moments, hazard function, inferential studies and an application in a real database are provided to illustrate the usefulness of the proposed model.

2   The Model Definition

If $G(x)$ denotes the cumulative distribution function (cdf) of a random variable X, then the generalized class of distributions, as defined by Eugene *et al.* (2002), can be define as

$$F(x) = I_{G(x)}(a, b), \tag{1}$$

for $a > 0$ e $b > 0$, where

$$I_{G(x)}(a, b) = \frac{B_{G(x)}(a, b)}{B(a, b)} \tag{2}$$

denotes the incomplete beta ration, and the incomplete beta function is given by

$$B_{G(x)}(a, b) = \int_0^{G(x)} w^{a-1}(1 - w)^{b-1}dw, \tag{3}$$

and

$$B(a, b) = \Gamma(a)\Gamma(b).$$

The probability density function corresponding to (1) is given by

$$f(x) = \frac{g(x)}{B(a, b)}(G(x))^{a-1}(1 - G(x))^{b-1}, \tag{4}$$

where $g(x)$ is the pdf corresponding to $G(x)$.

In this article, we consider the case when G(x) is the cumulative distribution function of the Chi-square distribution with parameter α. Thus, the random variable X follows the beta Chi-square distribution $BCHI(\alpha, a, b)$, with probability density function (pdf)

$$BCHI(\alpha, a, b) = \frac{x^{\alpha/2-1}e^{-x/2}}{2^{\alpha/2}\Gamma(\alpha/2)B(a, b)}\left(\frac{\gamma(\alpha/2, x/2)}{\Gamma(\alpha/2)}\right)^{a-1}\left(1 - \frac{\gamma(\alpha/2, x/2)}{\Gamma(\alpha/2)}\right)^{b-1}, \tag{5}$$

where $\gamma(\alpha/2, x/2) = \int_0^{x/2} t^{\alpha/2-1}exp(-t)dt$ is the incomplete Gamma function.

By using Equation (1), the cdf of the $BCHI(\alpha, a, b)$ is given by

$$F(x) = \frac{1}{B(a, b)}\int_0^{\frac{\gamma\left(\frac{\alpha}{2}, \frac{x}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}} w^{a-1}(1 - w)^{b-1}dw. \tag{6}$$

The BCHI distribution for $a = b = 1$ reduces to the Chi-square distribution with parameter α. The Fig. 1 illustrates the distribution function (6) and the density function (5), respectively, of BCHI, for different values of α.

3 General formula for moments

In statistical analysis, it is essential to study the moments, some of the most important characteristics of a distribution can be studied using moments, such as trend, dispersion, asymmetry and kurtosis. According to Cordeiro and Nadarajah
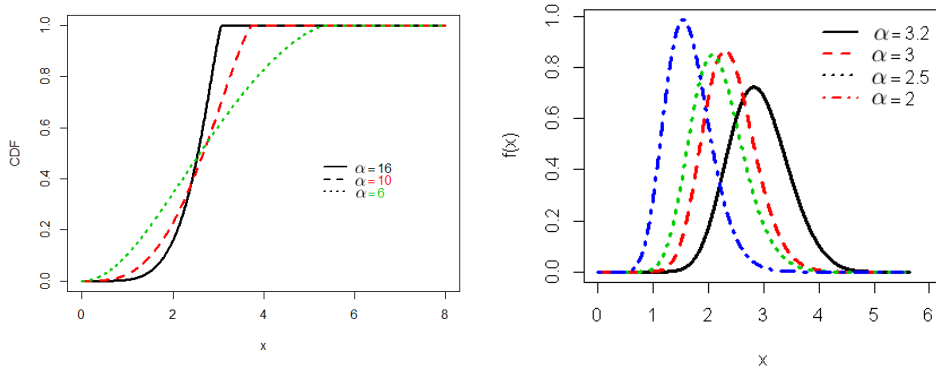
Figure 1 - Illustration of some possible forms of the distribution function (7) and the density function (6), for constant values of $a = 0.7$, $b = 1.35$ and different values of (α).

(2011), we must assume that X has pdf of any primitive G distribution, in our case the Chi-square distribution, and Y follows the Beta Chi-square distribution function. Here, we use power series to rewrite Chi-square beta distribution, as follows

$$\{1 - G(x)\}^{b-1} = \sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} G(x)^i.$$

Repeating this process, we can rewrite the equation (4) as follows

$$f(x) = g(x) \sum_{i,j=0}^{\infty} \sum_{r=0}^{j} w_{i,j,r} G(x)^r, \tag{7}$$

on what

$$w_{i,j,r} = \frac{(-1)^{1+j+r}}{B(a,b))} \binom{b-1}{i} \binom{a+i-1}{j} \binom{j}{r}.$$

More details can be found in Cordeiro and Nadarajah (2011). In order to find ordinary moments, we use the probability-weighted moments (PWM) method, proposed by Greenword, *et al.* (1979) and later applied to the Betas distributions by Cordeiro and Nadarajah (2011).

According to Greenword, *et al.* (1979), a distribution function $F = F(x) = P(X \leq x)$ can be characterized by PWM, when it is defined as

$$M_{s,r} = E[X^s F^r].$$

Consider that X has pdf of any G distribution function and Y follows the pdf of the beta G distribution. So, we have to $a \in \mathbb{Z}$

$$\mu_s' = \sum_{r=0}^{\infty} w_r M_{s,r+a-1}, \tag{8}$$

is for a real non integer,

$$\mu_s' = \sum_{r=0}^{\infty} w_r M_{s,r}. \tag{9}$$

3.1 Moments of the beta Chi-square

The sth moment of the Y can be expressed in terms of the (s,r)th PWM of X; i.e., $M_{s,r}$. We should still consider $\sum_{r=0}^{\infty} w_r = 1$ and $\sum_{l,j=0}^{\infty} \sum_{r=0}^{j} w_{l,j,r} = 1$. With the formulas above obtained by Cordeiro and Nadarajah (2011), we can find the moments of Beta Chi-square.

Supposing now that X has a Chi-square distribution with parameter $\frac{\alpha}{2} > 0$. We will get $M_{s,r}$ using series expansion for an incomplete gamma function, namely

$$\gamma(\alpha, x) = x^{\alpha} \sum_{m=0}^{\infty} \frac{(-x)^m}{(\alpha+m)m!} \tag{10}$$

making the variable change $\frac{\alpha}{2} = \lambda$ and applying an accumulated Chi-square distribution, it follows

$$G(x) = \left(\frac{x}{2}\right)^{\lambda} \sum_{m=0}^{\infty} \frac{\left(\frac{-x}{2}\right)^m}{(\lambda+m)m!}. \tag{11}$$

And then

$$M_{s,r} = \int_0^{\infty} \frac{x^{s+\lambda-1} exp(-\frac{x}{2})}{2^{\lambda}\Gamma(\lambda)} \left\{ \frac{\left(\frac{x}{2}\right)^{\lambda}}{\Gamma(\lambda)} \sum_{m=0}^{\infty} \frac{\left(\frac{-x}{2}\right)^m}{(\lambda+m)m!} \right\}^r dx, \tag{12}$$

replacing $\frac{x}{2} = u$ we have

$$\begin{aligned} M_{s,r} &= \int_0^{\infty} \frac{(2u)^{s+\lambda-1} exp(-u)}{2^{\lambda}\Gamma(\lambda)} \left\{ \frac{(u)^{\lambda}}{\Gamma(\lambda)} \sum_{m=0}^{\infty} \left\{ \frac{(-u)^m}{(\lambda+m)m!} \right\} \right\}^r 2du \\ &= \frac{2^s}{\Gamma(\lambda)^{r+1}} \int_0^{\infty} (u)^{s+\lambda-1} exp(-u) \left\{ (u)^{\lambda} \sum_{m=0}^{\infty} \left\{ \frac{(-u)^m}{(\lambda+m)m!} \right\} \right\}^r du. \end{aligned} \tag{13}$$

The integral in (13) can be obtained from the equations (24) e (25) Nadarajah (2008), how

$$M_{s,r} = \frac{2^s}{\Gamma(\lambda)^{r+1}} \left\{ \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r}}{(\lambda+m_1)\cdots(\lambda+m_r)m_1!\cdots m_r!} \right\} *$$

$$\int_0^{\infty} u^{s+\lambda(r+1)+m_1+\cdots+m_r^{-1}} exp(-u)du.$$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r}\Gamma(s+\lambda(r+1)+m_1+\cdots+m_r)}{(\lambda+m_1).\cdots.(\lambda+m_r)m_1!\cdots m_r!}$$

Using the Lauricella function of type A (EXTON, 1978), it follows that

$$M_{s,r} = \frac{2^s}{\Gamma(\lambda)^{r+1}}\Gamma(s+\lambda(r+1))F_A^{(r)}\left(s+\lambda(r+1);\lambda,\cdots,\lambda;\lambda+1,\cdots\lambda+1;-1,\cdots,-1\right). \quad (14)$$

Thus, the moments of the beta Chi-square distribution can be written as infinite weighted sums of the Lauricella function of type A, defined for integer and noninteger real number, replacing (14) in (8) and (9), respectively.

4 Hazard functions

Let X be a continuous random variable with distribution function F , and probability density function (pdf) f, then the hazard function is given by $h(x) = f(x)/(1 - F(x))$. The hazard function of the BCHI distribution is

$$h(x) = \frac{2^{-\alpha/2}e^{-x/2}x^{-1+\alpha/2}\left(\frac{\gamma(\alpha/2,x/2)}{\Gamma(\alpha/2)}\right)^{a-1}\left(1-\frac{\gamma(\alpha/2,x/2)}{\Gamma(\alpha/2)}\right)^{b-1}}{\left(B(a,b)-B(\frac{\gamma(\alpha/2,x/2)}{\Gamma(\alpha/2)},a,b)\right)\Gamma(\alpha/2)} \quad (15)$$

for $x \geq 0$, $\alpha > 0$, $a > 0$ and $b > 0$.

The plots at Fig. 2 show various shapes including monotonically decreasing, monotonically increasing with four combinations of values of the parameters. This flexibility makes the BCHI hazard rate function useful and suitable for behaviors which are more likely to be encountered or observed in the reality.

5 Inference

Let $x = (x_1, \cdots, x_n)$ be a random sample of the BCHI distribution with unknown parameter vector $\theta = (\alpha, a, b)$. The log likelihood $\ell = \ell(\theta; x)$ for θ is

$$\begin{aligned}
\ell &= \left(\frac{\alpha}{2}-1\right)\log(x) - \frac{x}{2} - \frac{\alpha}{2}\log(2) + (a-1)\log((\gamma(\frac{\alpha}{2},\frac{x}{2})) \\
&+ (b-1)\log\left(\Gamma\left(\frac{\alpha}{2}\right) - \gamma\left(\frac{\alpha}{2},\frac{x}{2}\right)\right) - (a+b-1)\log\left(\Gamma(\frac{\alpha}{2})\right) \\
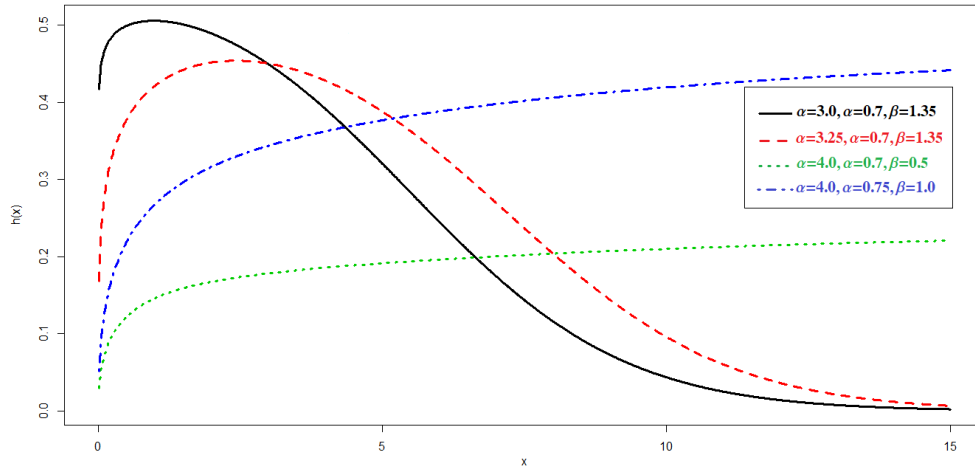&- \log(\beta(a,b))
\end{aligned} \quad (16)$$

Figure 2 - Plot Hazard Rate Function for selected parameters.

The maximum likelihood estimate (MLE) $\hat{\theta}$ of θ is calculated numerically from the nonlinear equations $U_\theta = 0$ using the EM algorithm. The components of the score vector $U_\theta = (U_\alpha, U_a, U_b)^T$, consider $\frac{\alpha}{2} = \lambda$, are given by

$$
\begin{aligned}
U_\lambda &= \sum_{i=1}^{n} \log(x_i) - n\log(2) + (a-1)\sum_{i=1}^{n} \frac{1}{\gamma(\lambda, \frac{x_i}{2})} \left\{ \gamma(\lambda, \frac{x_i}{2}) \log(\frac{x_i}{2}) \right. \\
&+ MeijerG[\{\{\}, \{1,1\}\}, \{\{0,0,\alpha\}, \{\}\}, \frac{x_i}{2}] \\
&+ (b-1)\sum_{i=1}^{n} \frac{1}{\gamma(\lambda, \frac{x_i}{2})} MeijerG[\ ] \left. \right\} - n(a+b-1)PolyGamma[0,\lambda],
\end{aligned}
$$

the MeijerG and PolyGamma functions were defined by Fields (1972) and Wolfram (1988), respectively.

$$
U_a = -n\log[\Gamma[\lambda]] + \sum_{i=1}^{n} \log[\gamma[\lambda, \frac{x_i}{2}]] - nPolyGamma[0,a] + nPolyGamma[0,a+b].
$$

$$
U_b = -n\log[\Gamma[\lambda]] + \sum_{i=1}^{n} \log[\Gamma[\lambda] - \gamma[\lambda, \frac{x_i}{2}]] - nPolyGamma[0,b] + nPolyGamma[0,a+b].
$$

The EML's $\hat{\theta} = (\hat{\lambda}, \hat{a}, \hat{b})^\top$ in $\theta = (\lambda, a, b)^\top$ are simultaneously the solutions of the equations $U_\lambda = U_a = U_b = 0$ and can be obtained numerically using the Newton-Raphson method. The observed information matrix can be obtained by $J_n(\boldsymbol{\theta}) = \frac{-\partial^2 \ell(\theta)}{\partial(\theta)\partial\theta^\top} = -U_{ij}$, for $i, j = \lambda, a$ and b,

$$
\begin{aligned}
U_{\lambda\lambda} &= \sum_{i=1}^{n} \{ \frac{-1}{\gamma(\lambda, \frac{x_i}{2})^2}(a-1)\left(\gamma(\lambda, \frac{x_i}{2}^2)\log(\frac{x_i}{2}) + MeijerG[\{\{\}, \{1,1\}, \{0,0,\lambda\}, \{\}\}, \frac{x_i}{2}]\right)^2 \\
&+ \frac{1}{\gamma(\lambda, \frac{x_i}{2})}(a-1)\left(\log(\frac{x_i}{2})MeijerG[\{\{\}, \{1,1\}, \{0,0,\lambda\}, \{\}\}, \frac{x_i}{2}]\right. \\
&+ \log\left(\frac{x_i}{2}\right)\left(\gamma(\lambda, \frac{x_i}{2})\log\left(\frac{x_i}{2}\right) + MeijerG[\{\{\}, \{1,1\}, \{0,0,\lambda\}, \{\}\}, \frac{x_i}{2}]\right) \\
&+ 2MeijerG[\{\{\}, \{1,1\}, \{0,0,\lambda\}, \{\}\}, \frac{x_i}{2}]) \} \\
&+ \sum_{i=1}^{n} \{ \frac{1-b}{\Gamma(\lambda) - \gamma(\lambda, \frac{x_i}{2})} \\
&\cdot \left(-\gamma(\lambda, \frac{x_i}{2})\log\left(\frac{x_i}{2}\right) - MeijerG[\{\{\}, \{1,1\}, \{0,0,\lambda\}, \{\}\}, \frac{x_i}{2}] + \Gamma(\lambda)\,PolyGamma(0,\lambda)\,\right)^2 \\
&+ \frac{-1+b}{\Gamma(\lambda + \gamma(\lambda, \frac{x_i}{2}))}\left(-\log(\frac{x_i}{2})MeijerG[\{\{\}, \{1,1\}, \{0,0,\lambda\}, \{\}\}, \frac{x_i}{2}] \right) \} \\
&- 2MeijerG[\{\{\}, \{1,1,1\}, \{0,0,0,\lambda\}, \{\}\}, \frac{x_i}{2}] + \Gamma(\lambda)PolyGamma(0,\lambda)^2 \\
&+ PolyGamma(1,\lambda)\,) \} + n(a+b-1)PolyGamma[1,\lambda].
\end{aligned}
$$

$$
\begin{aligned}
U_{aa} &= -nPolyGamma[1,a] + nPolyGamma[1, a+b]. \\
U_{bb} &= -nPolyGamma[1,b] + nPolyGamma[1, a+b]. \\
U_{\lambda a} &= \sum_{i=1}^{n} \frac{\gamma\left(\lambda, \frac{x_i}{2}\right)\log\left(\frac{x_i}{2}\right) + MeijerG[\{\{\}, \{1,1\}, \{0,0,\lambda\}, \{\}\}, \frac{x_i}{2}]}{\gamma\left(\lambda, \frac{x_i}{2}\right)} - nPolyGamma[0, \lambda]. \\
U_{\lambda b} &= \sum_{i=1}^{n} \frac{-\gamma\left(\lambda, \frac{x_i}{2}\right)\log\left(\frac{x_i}{2}\right) - MeijerG[\{\{\}, \{1,1\}, \{0,0,\lambda\}, \{\}\}, \frac{x_i}{2}]}{\Gamma(\lambda) - \gamma\left(\lambda, \frac{x_i}{2}\right)} - nPolyGamma[0, \lambda]. \\
U_{ab} &= 0.
\end{aligned}
$$

We can calculate the likelihood ratio (LR) test to test some sub-models of the BCHI distribution. For example, we can use **LR** to check whatever the fit using the BCHI distribution is statistically "best" than an fit using the χ^2 distribution, for a given data set. Consider the partition $\theta = \left(\theta_1^\top, \theta_2^\top\right)$ the vector of parameters of the BCHI distribution, where θ_1 is a subset of parameters of interest and θ_2 is a subset of perturbation parameter vectors. The **LR** statistic for testing null hypotheses $\mathcal{H}_0 : \theta_1 = \theta_1^{(0)}$ versus the alternative hypothesis $\mathcal{H}_1 : \theta_1 \neq \theta_1^{(0)}$ it is given by $w = 2\left\{\ell(\hat{\theta}) - \ell(\tilde{\theta})\right\}$, being $\hat{\theta}$ and $\tilde{\theta}$ are the MLE's under the null hypothesis

and the alternative, respectively, and $\theta_1^{(0)}$ is a specified parameter vector. The w statistic is asymptotically distributed $(n \to \infty)$ for χ_k^2, where k is the size of the subset of interest θ_1. Then, we can compare the BCHI model against the model χ^2 to test $\mathcal{H}_0 : a = b = 1$ versus $\mathcal{H}_1 : a \neq b \neq 1$ and the **LR** statistic becomes $w = 2\left\{\ell\left(\hat{\lambda}, \hat{a}, \hat{b}\right) - \ell\left(\tilde{\lambda}, 1, 1\right)\right\}$, where $\hat{\lambda}, \hat{a}$ and $\hat{b}$ they are the MLE's in $\mathcal{H}_1$ e $\tilde{\lambda}, \tilde{a}$ e $\tilde{b}$ are MLE's under $\mathcal{H}_0$.

Non-nested distributions can be compared based on the Akaike information criterion given by the formula $AIC = -2\ell(\hat{\theta}) + 2p$ and the Bayesian information criterion defined by $BIC = 2\ell(\hat{\theta}) + p\log(n)$, being p the number of parameters in the model. The lowest value distribution of any of these criteria (among all the distributions considered) is generally considered the best choice to describe the data set.

6 Application

In this section, the Beta Chi-square distribution was adjusted using a real database and then compared with the Chi-square distribution in order to compare them and verify their potentiality. The data set represents the survival times, in weeks, of 33 patients suffering from acute Myelogeneous Leukemia. These data have been analyzed by Feigl and Zelen (1965). The data set was recently studied by Woll *et al.* (2014) and Altun *et al.* (2021). Obtaining the maximum likelihood estimates (MLEs) for the distribution parameters, the maxLik function in maxLik-package of the statistical software R was used, and the iteration method was Newton Raphson. The estimated values of the parameters, the Hannan-Quinn information criterion (HQC), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) are presented in the Table 1.

The Figure 3 shows the fits of the BCHI and Chi-square. According to the illustration, the good fit of the BCHI distribution is observed.

Table 1 lists the MLEs of the models parameters BCHI and Chi-square, and the statistics AIC and BIC. These results show that the BCHI distribution has the lowest statistics and so it could be chosen as the best model.

Table 1 - The statistics HQC, AIC and BIC and estimates of the model parameters for the acute myelogeneous data

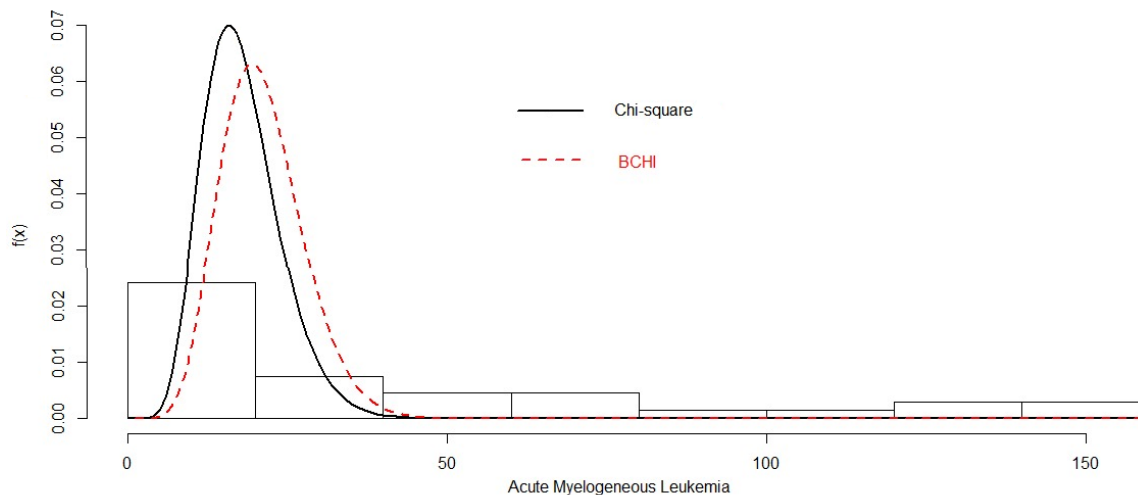| MODEL	HQC	AIC	BIC	α	a	b
BCHI	870.969	869.459	873.948	17	0.7	1.341
Chi-square	969.036	968.533	970.029	17.93		

Figure 3 - The histogram of the acute myelogeneous data and the estimated fitted distributions.

7   Conclusions

In this work, we define a new model called Beta Chi-square distribution. It is observed the new distribution of three parameters is quite similar in nature to the distribution Chi-square. Although the new distribution is more flexible due the fact it has more parameters than the primitive distribution.

Finally, we adjusted the BCHI model to a set of real data and observed that the new distribution performed better than the compared model, according to the statistics presented.

SACRAMENTO, K.P. N.; SACRAMENTO, V. P. Distribuição Beta Chi-quadrado propriedades e aplicações. *Braz. J. Biom.,* Lavras, v.40, n.1, p.108-119, 2022.

- *RESUMO: Neste artigo, apresentamos a distribuição de três parâmetros, denominada distribuição Beta Qui-quadrado (BCHI), que contém a distribuição Qui-quadrado como um submodelo. Sua função de densidade pode ser expressa como uma combinação linear da função de densidade qui-quadrado. Algumas propriedades estruturais desta distribuição, como os momentos e a função de risco são apresentados. As estimativas dos parâmetros do modelo são realizadas usando o método de Máxima Verossimilhança. Obtemos a matriz de informação observada e discutimos métodos de inferência. Para demonstrar a utilidade da distribuição, um conjunto de dados real é analisado.*

- *PALAVRAS-CHAVE: Momentos; Função de risco; Método de máxima verossimilhança; Matriz de informação observada.*

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