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ARTICLE

Likelihood Ratio Test For The Multivariate Normal Generalized Variance

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Abstract
An interesting measure of variability in multivariate populations is the determinant of the covariance matrix $\mathbf{\Sigma}_{p\times p}$, denoted as $|\mathbf{\Sigma}|$, commonly referred to as generalized variance. This measure succinctly captures the dispersion of a multivariate population into a single value, while accounting for inter-variable dependencies. Consequently, it finds applications across various domains concerned with assessing dispersion within multivariate populations of interest. In this study, we introduce a likelihood ratio test for the generalized variance of multivariate normal distributions, accompanied by a theoretical exposition on the distribution theory of sample generalized variances. We propose both the Likelihood Ratio Test (LRT) and the Bartlett-Corrected Likelihood Ratio Test (BCLRT) for assessing the hypothesis that the generalized variance equals a parameter η, where $\eta \in \mathbb{R}$. The development of these tests is purely theoretical. Our recommendation is to employ the BCLRT test primarily in scenarios where $p = 2$, particularly when $n > 30$. As for the LRT test, we suggest its application in cases where $p = 2$ or $p = 3$, provided that $n > 30$, and for $p = 5$ when $n > 50$.
Keywords: Monte Carlo; Standardized generalized variance; Variability measure.

1. Introduction

An interesting measure of variability in a multivariate population is the determinant of the $p \times p$ covariance matrix $\mathbf{\Sigma}$, denoted as $|\mathbf{\Sigma}|$, known as generalized variance. As highlighted by Najarzadeh (2019), this measure finds extensive usage across various domains, such as multivariate control chart analysis (Bersimis *et al.,* 2007; Djauhari, 2005; Djauhari *et al.,* 2008; Lee & Khoo, 2017; Noor & Djauhari, 2014; Yeh *et al.,* 2006, 2003), reliability modeling (Tallis & Light, 1968), signal processing (Bhandary, 1996), clustering (Gupta, 1982), optimal design (Pukelsheim, 2006), and optimal allocation in stratified sampling Arvanitis & Afonja, 1971. Generalized variance serves as a measure of the hypervolume occupied by the distribution of random variables in the p-dimensional space. Another noteworthy measure is the standardized generalized variances, allowing comparisons among sets of different dimensions, which is defined by the geometric means of the eigenvalues of $\boldsymbol{\Sigma}$, represented as $|\boldsymbol{\Sigma}|^{1/p}$ (Sengupta, 1987a,b).

Various procedures for constructing confidence intervals and hypothesis tests were described by Jafari & Kazemi (2014), who evaluated their performance through Monte Carlo simulations. Eaton (1967) demonstrated that the sample generalized variance has an monotone likelihood ratio property, consequently allowing for the derivation of a UMP invariant test. However, there appears to be a gap in the literature regarding the likelihood ratio test for the generalized variance of a normal population. Conversely, tests for comparisons and confidence intervals for the product of several (standardized) generalized variances are proposed by Najarzadeh (2017, 2019). Our focus is on constructing the likelihood ratio test (LRT) for the null hypothesis $H_0 : |\boldsymbol{\Sigma}| = \eta$, assuming multivariate normality.

Additionally, we provide a detailed step-by-step explanation of the LRT and present the theory of the distribution of the multivariate normal sample generalized variance developed thus far. Monte Carlo simulations were conducted to compare the performance of our test with others considered in this study. Finally, we illustrate the effectiveness of our method using real data.

2. Matherials and Methods

2.1 Normal sample generalized variance distribution

The theorem by Bartlett (1934) pertains to the transformation of Wishart matrices through the Cholesky decomposition. This result holds significant importance as it constitutes the most commonly employed method for generating realizations of Wishart random variables, denoted as $\boldsymbol{W}$ $(p \times p)$. Let $\boldsymbol{T}$ be an upper triangular matrix, which represents the Cholesky factor of $\boldsymbol{W}$, and consider the transformation $\boldsymbol{W} = \boldsymbol{T}^\top \boldsymbol{T}$. The subsequent result will be demonstrated directly utilizing the Wishart density. An alternative proof can be found in Kollo & von Rosen (2005).

Theorem 1 (Bartlett's theorem). *Let* $\boldsymbol{W} \sim W_p(\nu, \boldsymbol{I})$ $(\nu \geq p)$ *and* $\boldsymbol{W} = \boldsymbol{T}^\top \boldsymbol{T}$*, where* $\boldsymbol{T}$ *is an upper triangular matrix* $p \times p$ *with positive diagonal entries, then the elements* t_{ij} $(1 \leq i \leq j \leq p)$ *of* $\boldsymbol{T}$ *are independents and* $t_{ij} \sim N(0,1)$ $(1 \leq i < j \leq p)$ *and* $t_{ii}^2 \sim \chi^2_{\nu-i+1}$ $(i = 1, 2, \cdots, p)$.

Proof. The density of $\boldsymbol{W}$, when $\boldsymbol{\Sigma} = \boldsymbol{I}$ is given by

$$f_{W}(\boldsymbol{w}; n, \boldsymbol{I}) = \frac{|\boldsymbol{w}|^{(\nu-p-1)/2}}{2^{\nu p/2}\Gamma_p\left(\frac{\nu}{2}\right)} \exp\left\{-\frac{1}{2}\operatorname{tr}(\boldsymbol{w})\right\}, \tag{1}$$

where

$$\Gamma_p\left(\frac{\nu}{2}\right) = \pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma\left(\frac{\nu-i+1}{2}\right).$$

We should note that when using the Jacobian transformation method, we essentially have $p(p+1)/2$ variables in $\boldsymbol{W}$ (since it is symmetric). Therefore

$$\begin{aligned}\boldsymbol{W} =& \boldsymbol{T}^\top \boldsymbol{T}\\ =& \begin{bmatrix} T_{11} & 0 & 0 & \cdots & 0\\ T_{12} & T_{22} & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ T_{1p} & T_{2p} & T_{3p} & \cdots & T_{pp}\end{bmatrix}\begin{bmatrix} T_{11} & T_{12} & T_{13} & \cdots & T_{1p}\\ 0 & T_{22} & T_{23} & \cdots & T_{2p}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & T_{pp}\end{bmatrix}.\end{aligned}$$

Considering the vec operator and ignoring the lower triangular matrix elements of this product, we obtain

$$\text{vec}\left(\boldsymbol{T}^\top \boldsymbol{T}\right) = \begin{bmatrix} T_{11}^2 \\ T_{11}T_{12} \\ \vdots \\ T_{11}T_{1p} \\ T_{12}^2+T_{22}^2 \\ T_{12}T_{13}+T_{22}T_{23} \\ \vdots \\ T_{12}T_{1p}+T_{22}T_{2p} \\ \vdots \\ T_{1p}^2+T_{2p}^2+\cdots+T_{pp}^2 \end{bmatrix}.$$

Taking the first derivative in respect to vec($\boldsymbol{T}^\top$), the following $p(p+1)/2 \times p(p+1)/2$ lower triangular Jacobian matrix is obtained. This,

$$\frac{\partial\,\text{vec}\left(\boldsymbol{T}^\top \boldsymbol{T}\right)}{\partial\,\text{vec}\left(\boldsymbol{T}^\top\right)} = \begin{bmatrix} 2t_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ t_{12} & t_{11} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ t_{1p} & 0 & \cdots & t_{11} & 0 & \cdots & 0 \\ 0 & 2t_{12} & \cdots & 0 & 2t_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 2t_{pp} \end{bmatrix}.$$

This matrix has on its diagonal an element equal to $2t_{11}$ and other $p-1$ equal to t_{11}, an element equal to $2t_{22}$ and other $p-2$ equals t_{22} and so on. So the Jacobian of transformation is

$$J = 2t_{11}\prod_{i=1}^{p-1} t_{11} \times 2t_{22}\prod_{i=1}^{p-2} t_{22} \times 2t_{33}\prod_{i=1}^{p-3} t_{33} \times \cdots \times 2t_{pp} = 2^p \prod_{i=1}^{p} t_{ii}^{p-i+1}.$$

Also,

$$\text{tr}(\boldsymbol{w}) = \text{tr}(\boldsymbol{t}^\top \boldsymbol{t}) = \sum_{i\leq j}^{p} t_{ij}^2$$

and

$$|\boldsymbol{w}| = |\boldsymbol{t}^\top \boldsymbol{t}| = |\boldsymbol{t}|^2 = \prod_{i=1}^{p} t_{ii}^2.$$

Using the Jacobian transformation method, we obtain

$$
\begin{aligned}
f_{\boldsymbol{T}}(\boldsymbol{t};\nu) &= f_{\boldsymbol{W}}(\boldsymbol{w};n)|J| \\
&= \frac{\left(\prod_{i=1}^{p} t_{ii}^2\right)^{(\nu-p-1)/2}}{2^{\nu p/2}\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma\left(\frac{\nu-i+1}{2}\right)} \times \\
&\quad \times \exp\left\{-\frac{1}{2}\sum_{i\leq j}^{p} t_{ij}^2\right\} 2^p \prod_{i=1}^{p} t_{ii}^{p-i+1} \\
&= \frac{\prod_{i=1}^{p}\left(t_{ii}^{\nu-p-1}\, t_{ii}^{p-i+1}\right)}{2^{\nu p/2}2^{-p}\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma\left(\frac{\nu-i+1}{2}\right)} \prod_{i\leq j}^{p}\exp\left\{-\frac{t_{ij}^2}{2}\right\} \\
&= \frac{1}{2^{\nu p/2}2^{-p}2^{p/2-p/2}2^{-p(p-1)/4}} \prod_{i<j}^{p}\left[(2\pi)^{-1/2}e^{-t_{ij}^2/2}\right] \times \\
&\quad \times \prod_{i=1}^{p}\left[\frac{1}{\Gamma\left(\frac{\nu-i+1}{2}\right)}\left(t_{ii}^2\right)^{(\nu-i+1-1)/2} e^{-t_{ii}^2/2}\right] \\
&= \prod_{i<j}^{p}\left[(2\pi)^{-1/2}e^{-t_{ij}^2/2}\right] \times \\
&\quad \times \prod_{i=1}^{p}\left[\frac{1}{2^{(\nu-i+1)/2}\Gamma\left(\frac{\nu-i+1}{2}\right)}\left(t_{ii}^2\right)^{(\nu-i+1-1)/2} e^{-t_{ii}^2/2}\right],
\end{aligned}
$$

that correspond to the product of independent chi-square variables T_{ii}^2's with $\nu - i + 1$ degrees of freedom and standard normal variables T_{ij}'s, i.e., $N(0, 1)$. Since the joint probability density of the T_{ii}^2's and T_{ij}'s $(i < j)$ is the product of their marginal densities, they are independently distributed. □

The moment generating function of a chi-square variable X with ν degrees of freedom is given by $M_X(t) = (1 - 2t)^{-\nu/2}$ (Mittelhammer, 2013; Mood *et al.*, 1974). Therefore, the rth moment about the origin can be deduced, as shown in the following theorem.

Theorem 2 (Chi-square moments about the origin). *Let $X \sim \chi^2_\nu$ with $\nu > 0$ degrees of freedom and moment generating function of $M_X(t) = (1 - 2t)^{-\nu/2}$, then the rth moment about the origin from the distribution of X is*

$$\mathbb{E}\left[X^r\right] = \nu(\nu+2)(\nu+4)(\nu+6)\times\cdots\times(\nu+2r-2) = 2^r\frac{\Gamma\left(\frac{\nu}{2}+r\right)}{\Gamma\left(\frac{\nu}{2}\right)}. \quad (2)$$

Proof. We can observe that

$$M_X^{(1)}(t) = \frac{dM_X(t)}{dt} = \frac{d(1-2t)^{-\nu/2}}{dt} = \nu(1-2t)^{-\nu/2-1},$$

evaluating it at $t = 0$ results in

$$M_X^{(1)}(0) = \nu.$$

Repeating this procedure, to obtain the second derivative, we obtain

$$M_X^{(2)}(t) = \frac{d^2M_X(t)}{dt^2} = \frac{d\nu(1-2t)^{-\nu/2-1}}{dt} = \nu(\nu+2)(1-2t)^{-\nu/2-2},$$

that simplifies to

$$M_X^{(2)}(0) = \nu(\nu+2).$$

Similarly, for the third derivative, we have

$$M_X^{(3)}(t) = \frac{d^3M_X(t)}{dt^3} = \frac{d\nu(\nu+2)(1-2t)^{-\nu/2-2}}{dt} = \nu(\nu+2)(\nu+4)(1-2t)^{-\nu/2-3},$$

that gives

$$M_X^{(3)}(0) = \nu(\nu+2)(\nu+4).$$

Thus, repeating this procedure several times until the rth derivative, we arrive at the final result, which is given by

$$\begin{aligned} M_X^{(r)}(0) &= \nu(\nu+2)(\nu+4) \times \cdots \times (\nu+2r-2) \\ &= 2^r(\nu/2)(\nu/2+1)(\nu/2+2) \times \cdots \times (\nu/2+r-1) \\ &= = 2^r \frac{\Gamma\left(\frac{\nu}{2}+r\right)}{\Gamma\left(\frac{\nu}{2}\right)} = \mathbb{E}\left[X^r\right], \end{aligned}$$

by the gamma function properties. □

The generalized variance serves as a single-value summary of the covariance matrix, encompassing p variances and $p(p-1)/2$ covariances. It is defined by $|\boldsymbol{\Sigma}|$ for the population covariance matrix and $|S|$ for the sample covariance matrix. The determinant of the covariance matrix has a geometric interpretation, where the vectors of deviations of each observation from the mean form a hyperparallelogram in a p-dimensional space. The length of each side is proportional to the variance, and the angle between each pair of vectors is determined by a quantity proportional to the covariance (or correlation) between the two variables in the pair. The maximum volume is achieved when the angles are 90^0, and the variances of the p variables are equal. Thus, in the sample case with a sample size of n, we have $|S| = V^2(n-1)^{-p}$, where V represents the volume of this hyperparallelogram.

The generalized variance plays a significant role in the statistics of many multivariate hypothesis tests. While obtaining the probability density function of its exact distribution in the normal multivariate case is challenging in practice, Bartlett's theorem provides an effective means of determining its distribution. The next theorem is presented by Muirhead (1982).

Theorem 3 (Distribution of $|\boldsymbol{W}|$ from the Wishart distribution). *Let $\boldsymbol{W} \sim W_p(\nu, \boldsymbol{\Sigma})$ $(\nu \geq p)$, then the distribution of the random variable $|\boldsymbol{W}|/|\boldsymbol{\Sigma}|$ is the same of the $\prod_{i=1}^{p} \chi^2_{\nu-i+1}$, where $\chi^2_{\nu-i+1}$, $i = 1, 2, \cdots, p$ are independent random chi-square variables with $\nu - i + 1$ degrees of freedom, for $i = 1, 2, \cdots, p$.*

Proof. Consider that $|\boldsymbol{W}|/|\boldsymbol{\Sigma}| = |\boldsymbol{W}||\boldsymbol{\Sigma}|^{-1} = |\boldsymbol{\Sigma}|^{-1/2}|\boldsymbol{W}||\boldsymbol{\Sigma}|^{-1/2} = |\boldsymbol{\Sigma}^{-1/2}\boldsymbol{W}\boldsymbol{\Sigma}^{-1/2}|$. Since $\boldsymbol{W}$ follows a $W_p(\nu, \boldsymbol{\Sigma})$ distribution, by the linear transformation of Wishart random variables, $\boldsymbol{\Sigma}^{-1/2}\boldsymbol{W}\boldsymbol{\Sigma}^{-1/2}$ follows a $W_p(\nu, \boldsymbol{I}_p)$ distribution (Johnson & Wichern, 1998). Thus,

$$\boldsymbol{\Sigma}^{-1/2}\boldsymbol{W}\boldsymbol{\Sigma}^{-1/2} = \boldsymbol{T}^\top \boldsymbol{T},$$

where $\boldsymbol{T}$ is an upper triangular matrix. According to Bartlett's theorem 1, we have $|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{W}\boldsymbol{\Sigma}^{-1/2}| = \prod_{i=1}^{p} T_{ii}^2$, where T_{ii}^2 follows a chi-square distribution with $\nu - i + 1$ degrees of freedom. Additionally, by the same theorem, the T_{ii}^2 variables, for $i = 1, 2, \ldots, p$, are independent random variables. Hence,

$$|\boldsymbol{\Sigma}^{-1/2}\boldsymbol{W}\boldsymbol{\Sigma}^{-1/2}| = \prod_{i=1}^{p} T_{ii}^2 = \prod_{i=1}^{p} \chi^2_{\nu-i+1}.$$

as expected. □

We understand that the distribution of $|\boldsymbol{W}|/|\boldsymbol{\Sigma}|$ is the product of independent chi-square variables, which does not necessarily imply that we know its probability density function. As Muirhead (1982) points out, determining the probability density function in this case is not a straightforward task, despite the knowledge that it is the product of independent chi-square variables. This result holds significant importance, particularly when employing Monte Carlo simulation in inference processes. Moreover, it is crucial for determining various distributional properties, such as moments and asymptotic approximations.

Theorem 4 (Moments of $|\boldsymbol{W}|$). *Let $\boldsymbol{W} \sim W_p(\nu, \boldsymbol{\Sigma})$ $(\nu \geq p)$, then the rth $(r \geq 1)$ moment about the origin from the distribution of $|\boldsymbol{W}|$ is*

$$\mathbb{E}\left[|\boldsymbol{W}|^r\right] = |\boldsymbol{\Sigma}|^r \prod_{i=1}^{p} \left(\frac{2^r \Gamma[((\nu - i + 1)/2) + r]}{\Gamma[(\nu - i + 1)/2]} \right). \tag{3}$$

Proof. Considering that $|\boldsymbol{W}|/|\boldsymbol{\Sigma}|$ follows the distribution of the product of independent chi-square variables, then

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{|\boldsymbol{W}|}{|\boldsymbol{\Sigma}|}\right)^r\right] &= \mathbb{E}\left[\left(\prod_{i=1}^{p} \chi^2_{\nu-i+1}\right)^r\right] \\
&= \mathbb{E}\left[\prod_{i=1}^{p} \left(\chi^2_{\nu-i+1}\right)^r\right] \\
&= \prod_{i=1}^{p} \mathbb{E}\left[\left(\chi^2_{\nu-i+1}\right)^r\right] \qquad \text{(by the independence)} \\
&= \prod_{i=1}^{p} \frac{2^r \Gamma((\nu - i + 1)/2 + r)}{\Gamma((\nu - i + 1)/2)}, \qquad \text{(by theorem 2, expression (2))}.
\end{aligned}$$

But $\mathbb{E}\left[(|\boldsymbol{W}|/|\boldsymbol{\Sigma}|)^r\right]$ by the expectation linearity is $|\boldsymbol{\Sigma}|^{-r}\mathbb{E}\left[|\boldsymbol{W}|^r\right]$. Hence

$$\mathbb{E}\left[|\boldsymbol{W}|^r\right] = |\boldsymbol{\Sigma}|^r \prod_{i=1}^{p} \frac{2^r\Gamma((\nu-i+1)/2+r)}{\Gamma((\nu-i+1)/2)},$$

as pointed out. □

The mean and variance of the determinant of a Wishart matrix can be provided considering the results of theorem 4.

Corollary 4.1 (Mean and variance of $|\boldsymbol{W}|$). *Let $\boldsymbol{W} \sim W_p(\nu, \boldsymbol{\Sigma})$, then the mean and variance of $|\boldsymbol{W}|$ are, respectively, given by*

$$\mathbb{E}\left[|\boldsymbol{W}|\right] = |\boldsymbol{\Sigma}| \prod_{i=1}^{p}(\nu-i+1) \tag{4}$$

and

$$\mathbb{V}\left(|\boldsymbol{W}|\right) = |\boldsymbol{\Sigma}|^2 \prod_{i=1}^{p}(\nu-i+1)\left[\prod_{k=1}^{p}(\nu-k+3)-\prod_{k=1}^{p}(\nu-k+1)\right]. \tag{5}$$

Proof. For $r = 1$, utilizing (3), we have

$$\begin{aligned}
\mathbb{E}\left[|\boldsymbol{W}|\right] &= |\boldsymbol{\Sigma}|^1 \prod_{i=1}^{p} \frac{2^1\Gamma((\nu-i+1)/2+1)}{\Gamma((\nu-i+1)/2)} \\
&= |\boldsymbol{\Sigma}| \prod_{i=1}^{p} \frac{2(\nu-i+1)/2\Gamma((\nu-i+1)/2)}{\Gamma((\nu-i+1)/2)} \\
&= |\boldsymbol{\Sigma}| \prod_{i=1}^{p}(\nu-i+1).
\end{aligned}$$

Similarly, for $r = 2$, we have

$$\begin{aligned}
\mathbb{E}\left[|\boldsymbol{W}|^2\right] &= |\boldsymbol{\Sigma}|^2 \prod_{i=1}^{p} \frac{2^2\Gamma((\nu-i+1)/2+2)}{\Gamma((\nu-i+1)/2)} \\
&= |\boldsymbol{\Sigma}|^2 \prod_{i=1}^{p} \frac{2^2[(\nu-i+1)/2+1](\nu-i+1)/2\Gamma((\nu-i+1)/2)}{\Gamma((\nu-i+1)/2)} \\
&= |\boldsymbol{\Sigma}|^2 \prod_{i=1}^{p}(\nu-i+3)(\nu-i+1).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{V}\left(|\boldsymbol{W}|^2\right) =& \mathbb{E}\left[|\boldsymbol{W}|^2\right] - \mathbb{E}^2\left[|\boldsymbol{W}|\right]| \\
=& |\boldsymbol{\Sigma}|^2 \prod_{i=1}^{p}(\nu - i + 3)(\nu - i + 1) - \left[|\boldsymbol{\Sigma}| \prod_{i=1}^{p}(\nu - i + 1)\right]^2 \\
=& |\boldsymbol{\Sigma}|^2 \prod_{i=1}^{p}(\nu - i + 3)(\nu - i + 1) - |\boldsymbol{\Sigma}|^2 \prod_{i=1}^{p}(\nu - i + 1)^2 \\
=& |\boldsymbol{\Sigma}|^2 \prod_{i=1}^{p}(\nu - i + 1)\left[\prod_{k=1}^{p}(\nu - k + 3) - \prod_{k=1}^{p}(\nu - k + 1)\right],
\end{aligned}$$

as stated. □

Considering the sample covariance matrix S from a multivariate normal random sample of size n, we have $S = \boldsymbol{W}/\nu$, where $\nu = n - 1$, and the distribution of $|S|$ is the same as that of $|\boldsymbol{W}|\nu^{-p}$, utilizing determinant properties, where $\boldsymbol{W} \sim W_p(\nu, \boldsymbol{\Sigma})$. The distribution of S from a sample of the multivariate normal distribution is $W_p(\nu, \boldsymbol{\Sigma}\nu^{-1})$, as demonstrated in the following result.

Theorem 5 (Distribution of $|S|$). *Let $S \sim W_p(\nu, \boldsymbol{\Sigma}\nu^{-1})$ $(\nu \geq p)$, then $|S| \sim |\boldsymbol{\Sigma}|\nu^{-p} \prod_{i=1}^{p} \chi^2_{\nu-i+1}$, where $\chi^2_{\nu-i+1}$, $i = 1, 2, \cdots, p$ are independent random chi-square variables with degrees of freedom of $\nu - i + 1$ to the ith factor of the product.*

Proof. The proof is immediate, as $S \sim W_p(\nu, \boldsymbol{\Sigma}\nu^{-1})$. Therefore, by replacing $|\boldsymbol{\Sigma}|$ with $|\boldsymbol{\Sigma}\nu^{-1}|$ in theorem 3, and utilizing the fact that $|\boldsymbol{\Sigma}\nu^{-1}| = |\boldsymbol{\Sigma}|\nu^{-p}$, the result follows immediately. □

Anderson (2003) demonstrates the exact distribution for two particular cases,. The first case, for $p = 1$, is trivial, while the second case, for $p = 2$, results in the distribution of the product of two independent chi-square variables, which, according to the author, is also a chi-square distribution. The following corollary presents these two cases.

Corollary 5.1 (Distribution of $|S|$ for $p = 1$ and $p = 2$). *Let $S \sim W_p(\nu, \boldsymbol{\Sigma}\nu^{-1})$, then the distribution of $|S|$ for $p = 1$ is $\sigma^2\chi^2_\nu\nu^{-1}$ and for $p = 2$ is $|\boldsymbol{\Sigma}|\ (\chi^2_{2(\nu-1)})^2(4\nu^2)^{-1}$.*

Proof. For $p = 1$, according to theorem 5 we have $\boldsymbol{\Sigma} = \sigma^2$, $\nu^{-p} = \nu^{-1}$, and $\prod_{i=1}^{1} \chi^2_{\nu-i+1} = \chi^2_\nu$. Therefore, the result follows immediately, i.e., the distribution is scaled chi-square with ν degrees of freedom and a scale factor of σ^2/ν. The proof for $p = 2$ is left for the reader to consult the aforementioned author's publication. □

The exact distribution of $|S|$ is very complex to compute in real data, as it involves the distribution of products of independent chi-square variables. Therefore, it is advisable to employ some approximations for this distribution. An asymptotic normal approximation of this distribution is provided in Muirhead (1982) and Anderson (2003). The derivation of this approximation utilizes the delta method. Alternatively, Muirhead (1982) presents a method based on the characteristic function.

Theorem 6 (Asymptotic distribution of $|S|$). *Let $S \sim W_p(\nu, \boldsymbol{\Sigma}\nu^{-1})$ $(\nu \geq p)$, then $\sqrt{\frac{\nu}{2p}}\left(\frac{|S|}{|\boldsymbol{\Sigma}|} - 1\right)$ has an asymptotically standard normal distribution, $N(0,1)$.*

Proof. Considering the delta method, let $|S|/|\Sigma| = \prod_{i=1}^{p} \chi^2_{\nu-i+1}/\nu$, as shown in theorem 5. Note, by the properties of a chi-square variable, that $\mathbb{E}\left[\chi^2_{\nu-i+1}/\nu\right] = 1 - (i-1)/\nu$ and $\mathbb{V}\left(\chi^2_{\nu-i+1}/\nu\right) = 2\nu^{-1} - 2(i-1)\nu^{-2}$. Let $\chi^2_{\nu-i+1}$ denote the sum of squares of $\nu - i + 1$ standard normal variables $N(0,1)$, for $\nu \geq p$. By the central limit theorem, the asymptotic distribution of $\chi^2_{\nu-i+1}/\nu$ is approximately normal with mean 1 and variance $2\nu^{-1}$, since $(i-1)\nu^{-1}$ and $2(i-1)\nu^{-2}$ approach zero as $\nu \to \infty$. Considering the random vector, whose components are independently distributed as follows:

$$\boldsymbol{U} = \begin{bmatrix} \chi^2_{\nu}/\nu \\ \chi^2_{\nu-1}/\nu \\ \cdots \\ \chi^2_{\nu-p+1}/\nu \end{bmatrix},$$

we notice that $\boldsymbol{U}$ has an asymptotic multivariate normal distribution given by $N_p(\mathbf{1}_p, 2\nu^{-1}\boldsymbol{I})$.

Consider a real-valued function defined as $h(\boldsymbol{U}) = \prod_{i=1}^{p} U_i = \prod_{i=1}^{p} \chi^2_{\nu-i+1}/\nu = |S|/|\Sigma|$. Then, we obtain $\boldsymbol{h}'(\boldsymbol{u}) = [\prod_{j\neq i=1}^{p} U_j]_i$ $(p \times 1)$, for $i = 1, 2, \cdots, p$. Applying the delta method, we get

$$\mathbb{E}\left[h(\boldsymbol{U})\right] \simeq h(\boldsymbol{\mu}_{\boldsymbol{U}}) = 1$$

and

$$\mathbb{V}\left(h(\boldsymbol{U})\right) \simeq \boldsymbol{h}'^{\top}(\boldsymbol{\mu}_{\boldsymbol{U}})\boldsymbol{\Sigma}_{\boldsymbol{U}}\boldsymbol{h}'(\boldsymbol{\mu}_{\boldsymbol{U}}) = 2\nu^{-1}\mathbf{1}_p^{\top}\mathbf{1}_p = 2p\nu^{-1}.$$

Considering that U_i is asymptotically normal, the first-order approximation of $h(\boldsymbol{U})$ in the Taylor series will also be asymptotically normal. Consequently, the asymptotic distribution of $|S|/|\Sigma|$ is $N_1(1, 2p\nu^{-1})$. Additionally, we can deduce that $\sqrt{\nu}|S|/|\Sigma|$ has an asymptotic normal distribution $N_1(\sqrt{\nu}, 2p)$. Therefore, the desired result is immediately obtained using the transformation $\sqrt{\frac{\nu}{2p}}(|S|/|\Sigma| - 1)$. □

The normal approximation outlined in Theorem 5 is attributed to Anderson (2003). It is apparent that the random variable $|W|/|\Sigma|$ follows a distribution represented by the product of chi-square random variables. Thus,

$$U = \frac{\nu^p|S|}{|\Sigma|} = \frac{|W|}{|\Sigma|} \sim \prod_{i=1}^{p} \chi^2_{\nu-i+1}. \tag{6}$$

Two additional normal approximations of the U distribution are reported in the literature. One of these necessitates the following result:

$$\ln(\chi^2_{\nu}) \sim N\left(\psi(\nu/2) + \ln(2),\ \psi'(\nu/2)\right), \tag{7}$$

where $\psi()$ and $\psi'()$ represent the digamma (the logarithmic derivative of the gamma function) and trigamma (the first derivative of the digamma function) functions, respectively. Hence, let $Y = \ln(U) = p\ln(\nu) + \ln(|S|) - \ln(|\Sigma|)$, thus we obtain

$$Y \sim N\left(\sum_{i=1}^{p} \psi((\nu - p + 1)/2) + p\ln(2),\ \sum_{i=1}^{p} \psi'((\nu - p + 1)/2)\right). \tag{8}$$

This normal approximation is attributed to Sarkar (1989) and is recommended for $p > 3$.

The second normal approximation was developed by Djauhari (2009), and it is presented as:

$$|S| \sim N\left(b_1|\boldsymbol{\Sigma}|,\, b_2|\boldsymbol{\Sigma}|^2\right), \tag{9}$$

where

$$b_1 = \frac{1}{\nu^p}\prod_{i=1}^{p}(\nu - i + 1) \qquad \text{and} \qquad b_2 = \frac{b_1}{\nu^p}\prod_{i=1}^{p}(\nu - i + 3) - b_1^2.$$

2.2 Inferences on Normal sample generalized variance

Considering the exact and approximate distributions for some functions of the $|S|$ presented previously, we can consider the hypothesis tests for $|\boldsymbol{\Sigma}|$ from normal populations. The null and alternative hypotheses are

$$\begin{cases} H_0^{(a)}: |\boldsymbol{\Sigma}| = \eta & \text{against} \quad H_1^{(a)}: |\boldsymbol{\Sigma}| \neq \eta \\ H_0^{(b)}: |\boldsymbol{\Sigma}| \leq \eta & \text{against} \quad H_1^{(b)}: |\boldsymbol{\Sigma}| > \eta \\ H_0^{(c)}: |\boldsymbol{\Sigma}| \geq \eta & \text{against} \quad H_1^{(c)}: |\boldsymbol{\Sigma}| < \eta, \end{cases} \tag{10}$$

where $\eta > 0$ is a previously specified real value derived from some real problem of interest. For an exact test, a similar Monte Carlo version to the one proposed by Jafari & Kazemi (2014) was used. Thus, in the three hypothesis cases (10), we initially computed the value of the test statistic by

$$U_c = \frac{\nu^p|S|}{|\boldsymbol{\Sigma}_0|} = \frac{\nu^p|S|}{\eta}, \tag{11}$$

that under H_0 has distribution of $\prod_{i=1}^{p}\chi^2_{\nu-i+1}$, where $\nu = n-1$.

We considered a computational alternative to avoid overflow issues. Hence, the following Monte Carlo algorithm was used to obtain p-values for testing the above null hypotheses on $\boldsymbol{\Sigma}$:
Algorithm 1: Given p, n, and $|s|$:

1. Generate $\ln(U) = \sum_{i=1}^{p}\ln(\chi^2_{\nu-i+1})$, simulating $\chi^2_{\nu-i+1}$ for each $i = 1, 2, \cdots, p$.
2. Calculate $V_0 = p\ln(\nu) + \ln(|s|) - \ln(U)$ and $V = \exp(V_0)$.
3. Repeat steps $1-2$ for a large number os times, i.e., $m = 5000$ and obtain m values of V, denoting them by V_j, $j = 1, 2, \cdots, m$.
4. Calculate the $p$value for each case of (10), respectively, by

$$\begin{cases} q = \dfrac{1}{m}\displaystyle\sum_{j=1}^{m} I_{[0,\eta]}(V_j), & p\text{-value} = 2\min(q, 1-q) \quad \text{for } H_0^{(a)}, \\ q = \dfrac{1}{m}\displaystyle\sum_{j=1}^{m} I_{[0,\eta]}(V_j), & p\text{-value} = q \quad \text{for } H_0^{(b)}, \\ q = \dfrac{1}{m}\displaystyle\sum_{j=1}^{m} I_{[0,\eta]}(V_j), & p\text{-value} = 1-q \quad \text{for } H_0^{(c)}, \end{cases} \tag{12}$$

where $I_{[0,\eta]}(V_j)$ is the indicator function that é a função indicadora that returns 1 if $V \leq \eta$ and 0, otherwise.

We can also apply the test using any of the three normal approximations shown. We will start by presenting Anderson's approach (Anderson, 2003) in full detail. Subsequently, we will present only essential results for the other two approaches. Thus, in the case of the normal approximation of Anderson, for testing one of the three cases of the null hypothesis in (10), we initially compute the test statistic by

$$Z_c = \sqrt{\frac{\nu}{2p}}\left(\frac{|S|}{\eta} - 1\right), \tag{13}$$

where $\nu = n - 1$. The corresponding p-value depends on the hypothesis being tested. For $H_0^{(a)}$, we have

$$p\text{-value} = 2(1 - \Phi(|Z_c|)), \tag{14}$$

where $\Phi(x)$ is the cumulative distribution function of the standard normal distribution evaluated at x. For $H_0^{(b)}$ and $H_0^{(c)}$, we have

$$p\text{-value} = 1 - \Phi(Z_c) \qquad \text{and} \qquad p\text{-value} = \Phi(Z_c), \tag{15}$$

respectively. If the p-value was less or equal to the nominal significance level α, the null hypothesis H_0 should be rejected.

A similar approach was used for the Sarkar (1989)'s test, with $p > 3$. Initially, the test statistic, given by

$$Z_c = \frac{p\ln(\nu) + \ln(|S|) - \ln(\eta) - \mu_Y}{\sigma_Y} \tag{16}$$

should be computed, where $\mu_Y = \sum_{i=1}^{p} \psi((\nu - p + 1)/2) + p\ln(2)$ and $\sigma_Y^2 = \sum_{i=1}^{p} \psi'((\nu - p + 1)/2)$. The corresponding p-value depends on the null hypothesis being tested. For $H_0^{(a)}$, we have

$$p\text{-value} = 2(1 - \Phi(|Z_c|)) \tag{17}$$

where $\Phi(x)$ is the cumulative distribution function of the standard normal distribution evaluated at x. For $H_0^{(b)}$ and $H_0^{(c)}$, we have

$$p\text{-value} = 1 - \Phi(Z_c) \qquad \text{and} \qquad p\text{-value} = \Phi(Z_c), \tag{18}$$

respectively. If the p-value is less or equal to the nominal significance level α, the null hypothesis H_0 should be rejected.

Again for the Djauhari (2009)'s approach, the process is similar to the previous cases. The test statistic is given by

$$Z_c = \frac{|S| - b_1\eta}{\eta\sqrt{b_2}}, \tag{19}$$

where

$$b_1 = \frac{1}{\nu^p}\prod_{i=1}^{p}(\nu - i + 1) \qquad \text{and} \qquad b_2 = \frac{b_1}{\nu^p}\prod_{i=1}^{p}(\nu - i + 3) - b_1^2.$$

For $H_0^{(a)}$, the p-value is given by

$$p\text{-value} = 2(1 - \Phi(|Z_c|)). \tag{20}$$

For $H_0^{(b)}$ and $H_0^{(c)}$, the p-values are

$$p\text{-value} = 1 - \Phi(Z_c) \qquad \text{and} \qquad p\text{-value} = \Phi(Z_c), \tag{21}$$

respectively.

2.3 Likelihood ratio test on normal generalized variance

The likelihood ratio test for the null hypothesis H_0: $|\mathbf{\Sigma}| = \eta$, where $\eta > 0$, is developed in this study under multivariate normality. Let $\mathbf{X} \sim N_p(\boldsymbol{\mu}, \mathbf{\Sigma})$, and let $X1, X2, \ldots, Xn$ be a random sample from this distribution. It is known that the unrestricted maximum likelihood estimators of $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$ are well-established and given by $\bar{X}.$ and $\hat{\mathbf{\Sigma}} = n^{-1} \sum_{j=1}^{n} (Xj - \bar{X}.)(Xj - \bar{X}.)^\top$, respectively (Muirhead, 1982). Moreover, the unrestricted likelihood function is given by:

$$L_\Omega(X; \boldsymbol{\mu}, \mathbf{\Sigma}) = (2\pi)^{-np/2} |\mathbf{\Sigma}|^{-n/2} \exp\left\{ -\frac{1}{2} tr\left[\mathbf{\Sigma}^{-1} \left[\boldsymbol{W} + n\left(\bar{X}_. - \boldsymbol{\mu}\right)\left(\bar{X}_. - \boldsymbol{\mu}\right)^\top \right] \right] \right\}, \tag{22}$$

and its maximum is

$$L_\Omega(X; \hat{\boldsymbol{\mu}}, \hat{\mathbf{\Sigma}}) = (2\pi)^{-np/2} |\hat{\mathbf{\Sigma}}|^{-n/2} \exp\left\{ -\frac{np}{2} \right\}. \tag{23}$$

Under H_0, where $|\mathbf{\Sigma}| = \eta$, and denoting $\mathbf{\Sigma}$ by $\mathbf{\Sigma}_0$ to differentiate it from the unrestricted case, the restricted likelihood function is given by:

$$L_{\Omega_0}(X; \boldsymbol{\mu}, \mathbf{\Sigma}_0, \eta) = (2\pi)^{-np/2} \eta^{-n/2} \exp\left\{ -\frac{1}{2} tr\left[\mathbf{\Sigma}_0^{-1} \left[\boldsymbol{W} + n\left(\bar{X}_. - \boldsymbol{\mu}\right)\left(\bar{X}_. - \boldsymbol{\mu}\right)^\top \right] \right] \right\}. \tag{24}$$

The log-likelihood function is given by

$$g_{\Omega_0}(X; \boldsymbol{\mu}, \mathbf{\Sigma}_0, \eta) = -\frac{np}{2} \ln(2\pi) - \frac{n}{2} \ln(\eta) - \frac{1}{2} tr\left[\mathbf{\Sigma}_0^{-1} \left[\boldsymbol{W} + n\left(\bar{X}_. - \boldsymbol{\mu}\right)\left(\bar{X}_. - \boldsymbol{\mu}\right)^\top \right] \right]. \tag{25}$$

Taking the first derivative of the log-likelihood function (25) concerning $\boldsymbol{\mu}$ and equating it to zero, we will have the $\boldsymbol{\mu}$ estimator that maximizes (24), for $\mathbf{\Sigma}_0$ fixed. Therefore, the derivative is

$$\frac{\partial g_{\Omega_0}(X; \boldsymbol{\mu}, \mathbf{\Sigma}_0, \eta)}{\partial \boldsymbol{\mu}} = \mathbf{\Sigma}_0^{-1} \left(\bar{X}_. - \boldsymbol{\mu}\right),$$

where the solution when equated to $\mathbf{0}$ results in the maximum likelihood estimator given by

$$\hat{\boldsymbol{\mu}} = \frac{1}{n} \sum_{j=1}^{n} X_j = \bar{X}_. . \tag{26}$$

Therefore, the function

$$L_{\Omega_0}(X;\hat{\mu},\mathbf{\Sigma}_0,\eta)=(2\pi)^{-np/2}\eta^{-n/2}\exp\left\{-\frac{1}{2}\operatorname{tr}\left(\mathbf{\Sigma}_0^{-1}\boldsymbol{W}\right)\right\} \tag{27}$$

is such that $L_{\Omega_0}(X;\hat{\mu},\mathbf{\Sigma}_0,\eta)\geq L_{\Omega_0}(X;\mu,\mathbf{\Sigma}_0,\eta)$.

The corresponding log-likelihood function is

$$g_{\Omega_0}(X;\hat{\mu},\mathbf{\Sigma}_0,\eta)=-\frac{np}{2}\ln(2\pi)-\frac{n}{2}\ln(\eta)-\frac{1}{2}\operatorname{tr}\left(\mathbf{\Sigma}_0^{-1}\boldsymbol{W}\right). \tag{28}$$

We should maximize the function (27) or (28) in respect to the only remaining parameter, which is $\mathbf{\Sigma}_0$. This is equivalent to minimizing $\operatorname{tr}\left(\mathbf{\Sigma}_0^{-1}\boldsymbol{W}\right)$ subject to the restriction imposed by H_0 given by $|\mathbf{\Sigma}_0|=\eta$, $\eta>0$. Using Lagrange multipliers, we have the Lagrangian function, denoted by Φ and given by

$$\Phi(\mathbf{\Sigma}_0;\eta)=\operatorname{tr}\left(\mathbf{\Sigma}_0^{-1}\boldsymbol{W}\right)+\lambda\left(|\mathbf{\Sigma}_0|-\eta\right). \tag{29}$$

The partial derivatives by respect to $\mathbf{\Sigma}_0$ and λ are

$$\frac{\partial\Phi(\mathbf{\Sigma}_0;\eta)}{\partial\mathbf{\Sigma}_0}=-\mathbf{\Sigma}_0^{-1}\boldsymbol{W}\mathbf{\Sigma}_0^{-1}+\lambda|\mathbf{\Sigma}_0|\mathbf{\Sigma}_0^{-1}\quad\text{and}\quad\frac{\partial\Phi(\mathbf{\Sigma}_0;\eta)}{\partial\lambda}=|\mathbf{\Sigma}_0|-\eta,$$

that when equal to zero, we have from the second part that

$$|\hat{\mathbf{\Sigma}}_0|=\eta.$$

Replacing this result in the first part, we get

$$\begin{aligned}-\hat{\mathbf{\Sigma}}_0^{-1}\boldsymbol{W}\hat{\mathbf{\Sigma}}_0^{-1}+\lambda|\hat{\mathbf{\Sigma}}_0|\hat{\mathbf{\Sigma}}_0^{-1}&=\mathbf{0}\\ \lambda|\hat{\mathbf{\Sigma}}_0|\hat{\mathbf{\Sigma}}_0&=\boldsymbol{W}\quad\text{(after some algebra)}\\ \lambda\eta\hat{\mathbf{\Sigma}}_0&=\boldsymbol{W}\quad\text{(replacing }|\hat{\mathbf{\Sigma}}_0|=\eta\text{).}\end{aligned} \tag{30}$$

Taking the determinant on both sides of the last equation, we have

$$\lambda^p\eta^p|\hat{\mathbf{\Sigma}}_0|=|\boldsymbol{W}|=n^p|\hat{\mathbf{\Sigma}}|,$$

that results in

$$\begin{aligned}\lambda&=\frac{\sqrt[p]{|\boldsymbol{W}|}}{\eta^{(p+1)/p}}\quad\text{(replacing }|\hat{\mathbf{\Sigma}}_0|=\eta\text{)}\\ &=\frac{n\sqrt[p]{|\hat{\mathbf{\Sigma}}|}}{\eta^{(p+1)/p}}.\end{aligned}$$

Replacing this solution of λ in (30), we get

$$\frac{n\sqrt[p]{|\hat{\mathbf{\Sigma}}|}}{\eta^{(p+1)/p}}\eta\hat{\mathbf{\Sigma}}_0=n\hat{\mathbf{\Sigma}}=\boldsymbol{W},$$

resulting in the maximum likelihood estimator of $\boldsymbol{\Sigma}_0$, given by

$$\hat{\boldsymbol{\Sigma}}_0 = \frac{\sqrt[p]{\eta}}{\sqrt[p]{|\hat{\boldsymbol{\Sigma}}|}}\hat{\boldsymbol{\Sigma}} = \frac{\sqrt[p]{\eta}}{\sqrt[p]{|\boldsymbol{W}|}}\boldsymbol{W}. \tag{31}$$

Therefore, the maximum of the restricted likelihood function is

$$\begin{aligned} L_{\Omega_0}(\boldsymbol{X};\hat{\boldsymbol{\mu}},\hat{\boldsymbol{\Sigma}}_0,\eta) &= (2\pi)^{-np/2}\eta^{-n/2}\exp\left\{-\frac{p\sqrt[p]{|\boldsymbol{W}|}}{2\sqrt[p]{\eta}}\right\} \\ &= (2\pi)^{-np/2}\eta^{-n/2}\exp\left\{-\frac{np\sqrt[p]{|\hat{\boldsymbol{\Sigma}}|}}{2\sqrt[p]{\eta}}\right\}. \end{aligned} \tag{32}$$

The likelihood ratio test statistic is given by

$$\Lambda = \frac{L_{\Omega_0}(\boldsymbol{X};\hat{\boldsymbol{\mu}},\hat{\boldsymbol{\Sigma}}_0,\eta)}{L_{\Omega}(\boldsymbol{X};\hat{\boldsymbol{\mu}},\hat{\boldsymbol{\Sigma}})} = \left(\frac{\eta}{|\hat{\boldsymbol{\Sigma}}|}\right)^{-n/2}\exp\left\{-\frac{np}{2}\left[\sqrt[p]{\frac{|\hat{\boldsymbol{\Sigma}}|}{\eta}}-1\right]\right\}. \tag{33}$$

In the unrestricted model, we have a dimension given by $p + p(p+1)/2$ and in the constrained model we have a dimension given by $p + p(p+1)/2 - 1$. Therefore, under H_0: $|\boldsymbol{\Sigma}| = \eta$, we have, using the general theory of likelihood ratio tests, that $-2\ln(\Lambda)$, given by

$$\chi^2_c = n\left[\ln(\eta) - \ln(|\hat{\boldsymbol{\Sigma}}|)\right] + np\left[\sqrt[p]{\frac{|\hat{\boldsymbol{\Sigma}}|}{\eta}}-1\right],$$

has asymptotic chi-square distribution with $\nu = 1$ degree of freedom.

Example 1. *Use the example data where six hematological variables were measured at $n = 103$ individuals Jafari & Kazemi, 2014; Royston, 1983 and apply the test for the null hypothesis H_0: $|\boldsymbol{\Sigma}| = 6.0$ against H_1: $|\boldsymbol{\Sigma}| \neq 6.0$ considering a confidence coefficient of 95%. The sample estimate was $|s| = 6.2453$.*

The unrestricted maximum likelihood estimate of $|\boldsymbol{\Sigma}|$ is obtained by

$$|\hat{\boldsymbol{\Sigma}}| = \frac{(n-1)^p}{n^p}|s| = 5.890213.$$

Thus, the test statistic is

$$\begin{aligned} \chi^2_c &= n\left[\ln(\eta) - \ln(|\hat{\boldsymbol{\Sigma}}|)\right] + np\left[\sqrt[p]{\frac{|\hat{\boldsymbol{\Sigma}}|}{\eta}}-1\right] \\ &= 103\left[\ln(6) - \ln(5.890213)\right] + 103 \times 6 \times \left[\sqrt[6]{\frac{5.890213}{6}}-1\right] = 0.00292, \end{aligned}$$

whose p-value is 0.9569, which leads to non-rejection of H_0 at the 5% significance nominal level.

The tests outlined in Section 2.2 provide the same conclusions and showcase the following results:

- *Monte Carlo exact test: p-value = 0.474.*
- *Anderson Test: $Z_c = 0.11919$ and p-value = 0.9051.*
- *Sarkar Test: $Z_c = 0.7172$ and p-value = 0.47324.*

- *Djauhari Test:* $Z_c = 0.5869$ *and p-value* $= 0.55724$.

By the general theory of LRTs, this testing procedure is not appropriate for small sample sizes. Najarzadeh (2017) points out the standard approach to this problem, which consists of modifying the LRT statistic using Bartlett's correction. In this approach, to adjust the LRT, we use statistics $-2\phi\ln(\Lambda)$, where $\phi = \frac{n-1}{\mathbb{E}[-2\ln(\Lambda)]}$ is Bartlett correction factor. Using the 4th order Taylor polynomial approximation on the functions $|S|^{1/p}$ and $\ln(|S|)$ centered about the point $\mathbb{E}[|S|]$, we find the following results

$$\begin{aligned}
\mathbb{E}\left[-2\ln(\Lambda)\right] &= \mathbb{E}\left[n\ln(\eta) - n\ln(|\hat{\Sigma}|) + np\left(\frac{|\hat{\Sigma}|}{\eta}\right)^{1/p} - np\right] \\
&= n\ln(\eta) - n\mathbb{E}\left[\ln(|\hat{\Sigma}|)\right] + \frac{np}{\eta^{1/p}}\mathbb{E}\left[|\hat{\Sigma}|^{1/p}\right] - np \\
&= n\ln(\eta) - np + \frac{np}{\eta^{1/p}}\mathbb{E}\left[|\hat{\Sigma}|^{1/p}\right] - n\mathbb{E}\left[\ln(|\hat{\Sigma}|)\right] \\
&= n\ln(\eta) - np + \frac{np}{\eta^{1/p}}\mathbb{E}\left[\left(\frac{(n-1)^p}{n^p}|S|\right)^{1/p}\right] - n\mathbb{E}\left[\ln\left(\frac{(n-1)^p}{n^p}|S|\right)\right] \\
&= n\ln(\eta) - np + \frac{np}{\eta^{1/p}}\frac{(n-1)}{n}\mathbb{E}\left[|S|^{1/p}\right] - np\ln\left(\frac{n-1}{n}\right) - n\mathbb{E}\left[\ln(|S|)\right],
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{E}\left[|S|^{1/p}\right] \simeq{}& (\mathbb{E}[|S|])^{\frac{1}{p}} + \frac{1}{2p}\left(\frac{1}{p}-1\right)\mathbb{E}[|S|]^{\frac{1}{p}-2}\,\mathbb{V}(|S|) + \\
&+ \frac{1}{6p}\left(\frac{1}{p}-1\right)\left(\frac{1}{p}-2\right)\mathbb{E}[|S|]^{\frac{1}{p}-3}\,\mathbb{E}\left[(|S|-E|S|)^3\right] \\
&+ \frac{1}{24p}\left(\frac{1}{p}-1\right)\left(\frac{1}{p}-2\right)\left(\frac{1}{p}-3\right)\mathbb{E}[|S|]^{\frac{1}{p}-4}\,\mathbb{E}\left[(|S|-E|S|)^4\right]
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[\ln(|S|)] \simeq{}& \ln(\mathbb{E}[|S|]) - \frac{\mathbb{V}(|S|)}{2(\mathbb{E}[|S|])^2} + \frac{2}{6(\mathbb{E}[|S|])^3}\mathbb{E}\left[(|S|-\mathbb{E}[|S|])^3\right] - \\
&- \frac{6}{24(\mathbb{E}[|S|])^4}\mathbb{E}\left[(|S|-\mathbb{E}[|S|])^4\right]
\end{aligned}$$

Here, we can use the fact that $\mathbb{E}[|S|] = \dfrac{\mathbb{E}[|W|]}{\gamma^p}$ and the theorem 4 to calculate $\mathbb{E}[|W|^r]$, $r \geq 1$. Furthermore, all the central moments needed to find Taylor's approximation were expanded to non-central moments in which it was possible to apply theorem 4 directly. Thus, we can find Bartlett's correction $\phi = \dfrac{n-1}{\mathbb{E}\left[-2\ln(\Lambda)\right]}$. Therefore, we reject the null hypothesis of H_0 at the nominal significance level α, if the value of the modified Bartlett correction statistic $-2\phi\ln(\Lambda)$, is greater than the upper-tail α critical value of the chi-square distribution with 1 degrees of freedom.

3. Results and Discussion

3.1 Monte Carlo performance evaluation

Monte Carlo simulations were used to compare the actual sizes and powers of the following tests: i) Monte Carlo exact test (MCET) using algorithm 1, with $m = 2000$ replications, ii) Anderson test (AT), iii) Sarkar test (ST), iv) Djauhari test (DT), v) the proposed likelihood ratio test (LRT), and vi) Bartlett corrected likelihood ratio test (BCLRT). The two-sided hypothesis

$$H_0^{(a)}: |\boldsymbol{\Sigma}| = \eta \qquad \text{against} \qquad H_1^{(a)}: |\boldsymbol{\Sigma}| \neq \eta$$

were considered values of η, as 0.2 and 1.0, sample sizes n (15, 30, 50) and dimensions p (2, 3, 5, 10). Also, they were considered in 10000 Monte Carlo replications, in the same cases of Jafari & Kazemi (2014). Samples of size n and dimension p were generated from the multivariate normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}$. The six tests were applied in each case at the nominal significance level of α (0.01, 0.05 0.10). In the two cases concerning η (0.2 and 1), the samples were generated from multivariate normal populations with actual $|\boldsymbol{\Sigma}|$ ranging from 0.01 to 5.00 with a step size of 0.05. The empirical test powers and sizes were computed in each configuration for each set of 10000 simulations.

The evaluation of the tests will be performed graphically, more precisely through the empirical graph of the power function for each evaluated test. Three figures will be presented, each containing 4 graphs. In the first figure, the sample for the test is $n = 15$, with the number of p variables chosen being 2, 3, 5 and 10. The level of significance was set at 5%. $N = 10{,}000$ normal p-varied samples were generated for each p described and for each covariance matrix $\boldsymbol{\Sigma}$ such that their $|\boldsymbol{\Sigma}|$ values form a sequence between the numbers 0.01 and 5.00 in increments of 0.05. All analysis was built using *R software* (R CORE TEAM, 2019).

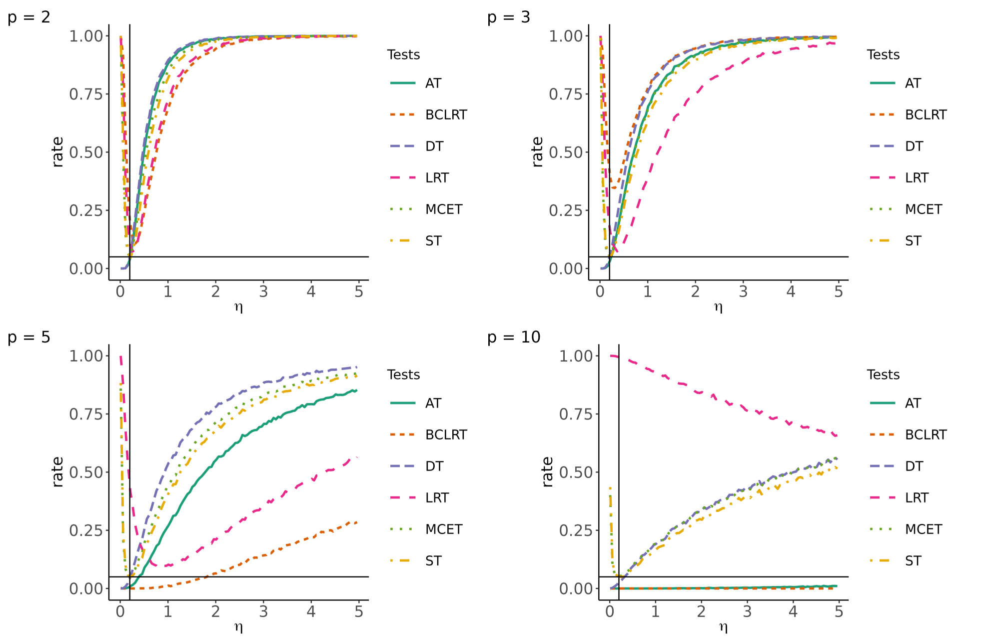

Figure 1. Performance evaluation of hypothesis tests at the 5% significance level for $\eta = 0.2$, $n = 15$ and $p = 2, 3, 5$ and 10.

Under H_0, $|\boldsymbol{\Sigma}| = 0.2$ was considered, so the type I error rate will be estimated when the sample of the normal p-varied is generated from a normal p-varied with covariance matrix, whose determinant is equal to 0.2. In Figure 1, we show the performance of the tests when the sample is of size 15 and $\eta = 0.2$.

It is evident from Figure 1 that when $p = 2$, all tests effectively control the type I error rate. Additionally, the power functions of all tests exhibit similar behavior. Concerning power, the AT and DT tests demonstrate superior performance compared to others, whereas the proposed LRT and BCLRT tests generally exhibit lower power across various scenarios.

In the case of p other than 2, the BCLRT test significantly deviated from controlling the type I error rate. Consequently, while considering the test's power, it should not be prioritized. Notably, the LRT test for $p = 3$ and $p = 5$ variables can be deemed liberal, as its estimated type I error rate surpasses the 5% significance level. However, despite its liberal nature, the LRT test generally exhibits lower power compared to alternative tests.

In scenarios where $p = 10$, the LRT test failed to control the type I error rate, whereas the AT and BCLRT tests consistently maintained it close to zero across all situations. Despite not displaying high power, the DT, MCET, and ST tests effectively controlled the type I error rate, with ST exhibiting slightly lower power compared to other tests.

Figure 2 illustrates the test performance for a sample size of 30 and $\eta = 0.2$. When $p = 2$ and $p = 3$, all tests successfully controlled the type I error rate, and their power functions exhibited similar behavior. Notably, the DT and MCET tests demonstrated superior power compared to others, while the proposed LRT and BCLRT tests generally exhibited lower power across various scenarios. However, for other values of p, the BCLRT test notably failed to control the type I error rate, and the LRT test for $p = 3$ and $p = 5$ variables can be considered liberal due to its estimated type I error rate exceeding the 5% significance level.

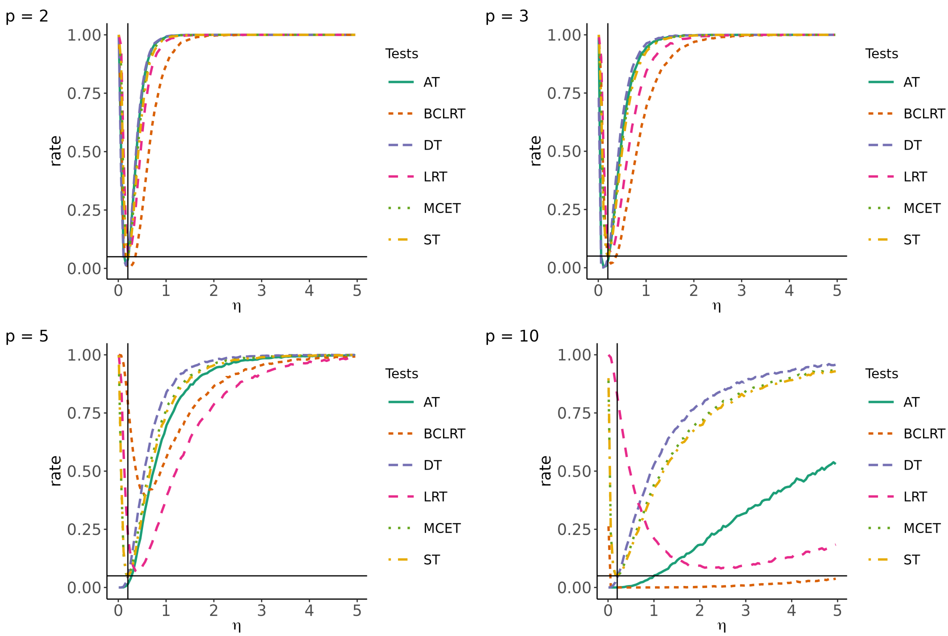

Figure 2. Performance evaluation of hypothesis tests at the 5% significance level for $\eta = 0.2$, $n = 30$ and $p = 2, 3, 5$ and 10.

For $p = 10$, the LRT test failed to control the type I error rate, while the AT and BCLRT tests consistently maintained it close to zero across all scenarios. Despite not exhibiting high power, the DT, MCET, and ST tests effectively controlled the type I error rate, with ST displaying slightly lower power compared to the other tests. Figure 3 depicts the performance of the tests for a sample size of 50 and η = 0.2.

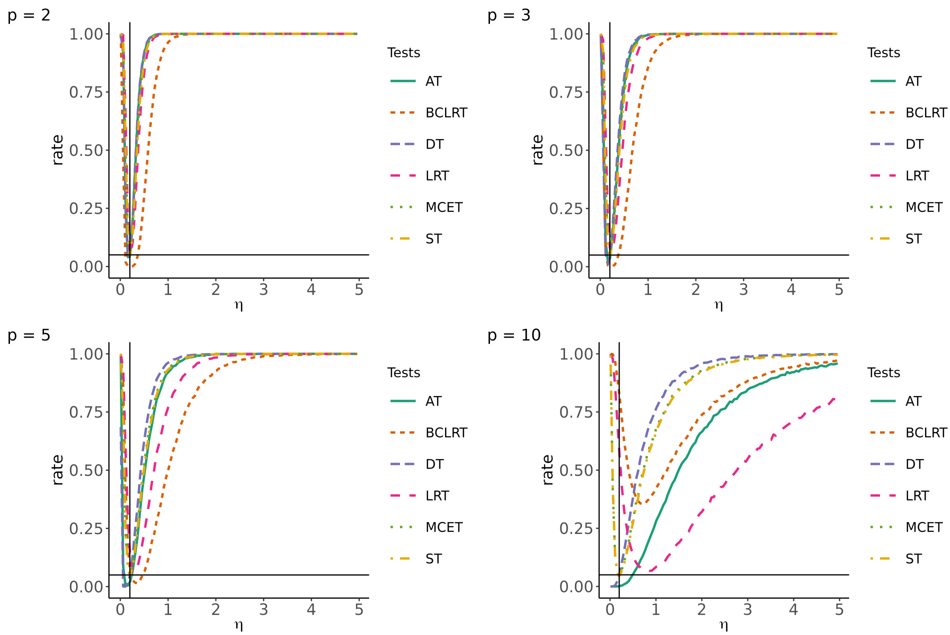

Figure 3. Performance evaluation of hypothesis tests at the 5% significance level for $\eta = 0.2$, $n = 50$ and $p = 2, 3, 5$ and 10.

For $p = 2$, 3, and 5, all tests effectively controlled the type I error rate, and their power functions exhibited similar behavior. Notably, the DT and MCET tests outperformed others in terms of power, while the proposed LRT and BCLRT tests generally exhibited lower power across most scenarios. However, for $p = 10$, both the LRT and BCLRT tests failed to control the type I error rate. In terms of power, the DT, ST, and MCET tests yielded the most favorable results.

Under the null hypothesis H_0, where $|\boldsymbol{\Sigma}| = 1$, the type I error rate was estimated by generating samples from a multivariate normal distribution with a covariance matrix determinant equal to 1.

Figure 4 illustrates the test performance for a sample size of 15 and $\eta = 1$. For $p = 2$, both the LRT and BCLRT tests may be deemed liberal, while the ST and MCET tests effectively controlled the type I error rate. When $p = 3$, the BCLRT test failed to control the type I error rate, and the LRT test exhibited liberal behavior, with the ST and MCET tests achieving the best performance. Regarding $p = 5$, the AT, LRT, and BCLRT tests performed poorly. Among the tests considered for $p = 10$, only the ST and MCET tests managed to control the type I error rate, although their power was not particularly high.

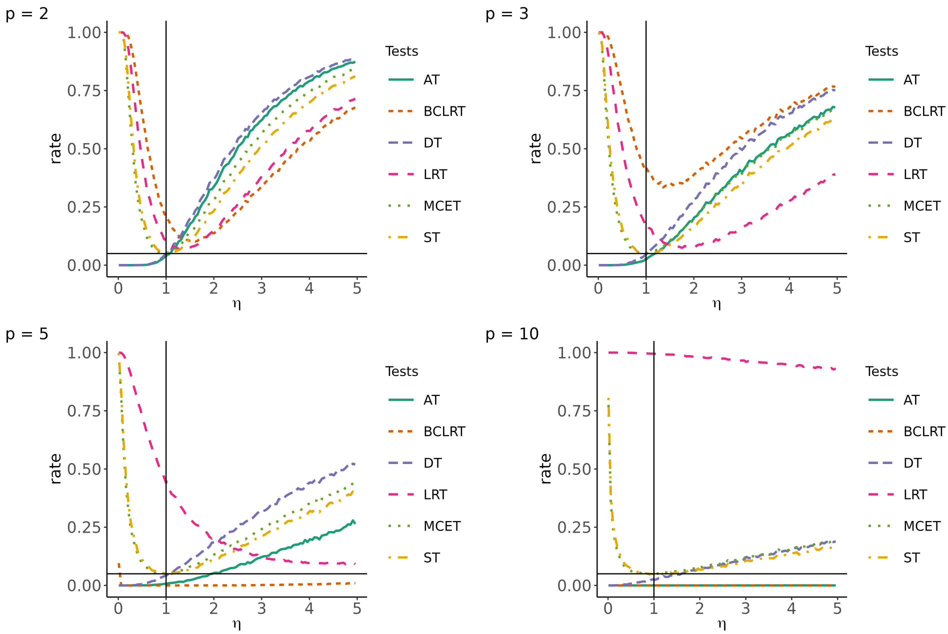

Figure 4. Performance evaluation of hypothesis tests at the 5% significance level for $\eta = 1$, $n = 15$ and $p = 2, 3, 5$ and 10.

Figure 5 illustrates the performance of the tests for a sample size of 30 and $\eta = 1$. For p equal to 2 and 3, the ST and MCET tests effectively controlled the type I error rate, while the AT, DT, and BCLRT tests were conservative, and the LRT test exhibited liberal behavior. Among these, the AT, DT, and MCET tests yielded the best results in terms of power. In the case of $p = 5$, the DT test displayed higher power, and the ST and MCET tests successfully controlled the type I error rate. However, for $p = 10$, the AT, LRT, and BCLRT tests performed worse than the others.

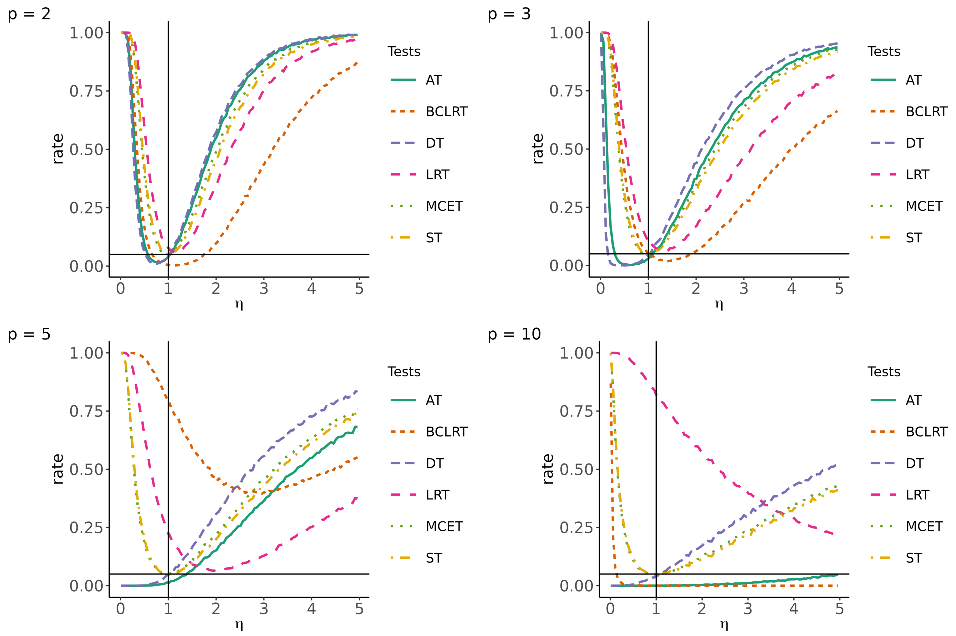

Figure 5. Performance evaluation of hypothesis tests at the 5% significance level for $\eta = 1$, $n = 30$ and $p = 2, 3, 5$ and 10.

Figure 6 depicts the performance of the tests for a sample size of 50 and $\eta = 1$. For p equal to 2, 3, and 5, the ST and MCET tests effectively controlled the type I error rate, while the AT, DT, and BCLRT tests exhibited conservative behavior, and the LRT test was liberal. In the case of $p = 10$, the ST and MCET tests outperformed others.

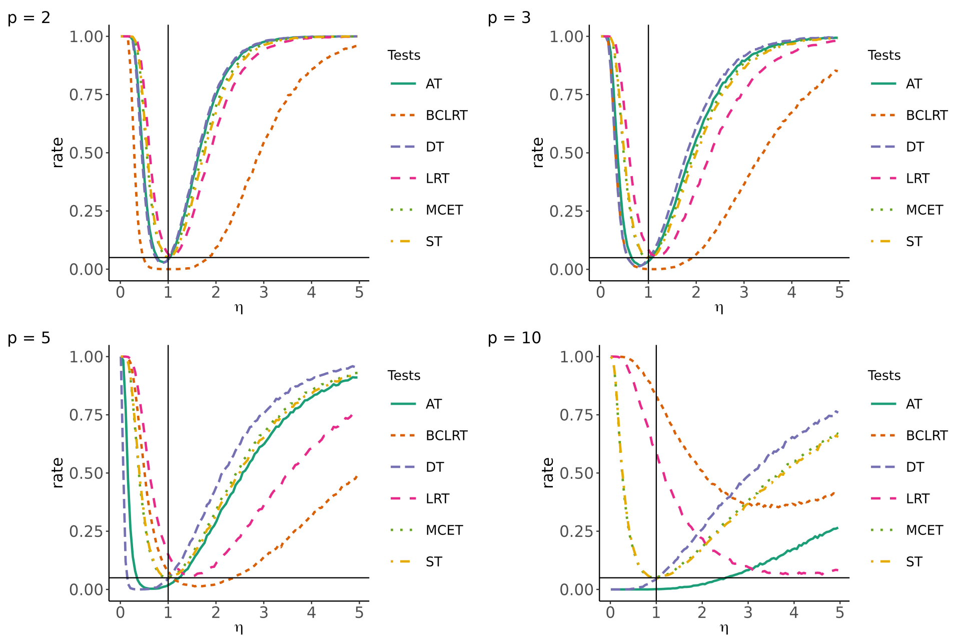

Figure 6. Performance evaluation of hypothesis tests at the 5% significance level for $\eta = 1$, $n = 50$ and $p = 2, 3, 5$ and 10.

The remaining results, along with details on test implementation and evaluation, can be found in the supplementary materials.

4. Conclusions

We proposed the LRT and BCLRT tests to examine the hypothesis that the generalized variance equals a parameter η, where $\eta \in \mathbb{R}$. The development of these tests was purely theoretical.

However, as the number of variables p increases, both the LRT and BCLRT tests fail to control the type I error rate adequately and exhibit low power. They also perform inferiorly compared to existing tests in the literature, particularly the ST and MCET tests. Nevertheless, the LRT test effectively controls the type I error rate for $p = 2$ and $p = 3$, and demonstrates good power performance even with small sample sizes. It also maintains type I error rate control for $p = 5$ when $n \geq 50$. On the other hand, the BCLRT test performs well only for $p = 2$. Both tests show improved performance with increasing sample size.

Therefore, we recommend using the BCLRT test primarily for scenarios where $p = 2$, especially when $n > 30$. As for the LRT test, we suggest its application in situations where $p = 2$ and $p = 3$ for $n > 30$, and $p = 5$ when $n > 50$.

Acknowledgments

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Complementary material

Script used in R

```
# Exact test for the normal generalized variance - MCET
# H_0: |Sig| = Delta_0 and confidence interval
exactGV.Test <- function(Delta_0, n, p, detS, alpha = 0.05,
                         alternative="two.sided", m = 5000)
{
j <- 1:p
nu <- n - j
nullDist <- function(n, p, nu)
{
lnU <- sum(log(rchisq(p, nu)))
V <- p * log(n - 1) - lnU + log(detS)
return(exp(V))
}
V <- matrix(n, m, 1)
V <- apply(V, 1, nullDist, p, nu)
#hist(V)
p.value <- length(V[V <= Delta_0]) / m
if (alternative == "two.sided") {
p.value <- 2 * min(p.value, 1 - p.value)
LL <- quantile(V, alpha / 2)
UL <- quantile(V, 1 - alpha / 2)
} else
if (alternative == "less") {
p.value <- 1 - p.value
LL <- 0
UL <- quantile(V, 1 - alpha)
} else
if (alternative == "greater") {
LL <- quantile(V, alpha)
UL <- Inf
}
return(list(p.value = p.value, LL = LL, UL = UL))
}



# AT
# Anderson  approximation test for the normal generalized variance
# H_0: |Sig| = Delta_0 and approximated confidence interval
andersonGV.Test <- function(Delta_0, n, p, detS, alpha = 0.05,
alternative="two.sided")
{
Zc <- sqrt((n - 1) /(2 * p)) * (detS / Delta_0 - 1)
if (alternative == "two.sided") p.value <- 2 * (1 - pnorm(abs(Zc)))
else
p.value <- pnorm(Zc, lower.tail = (alternative == "less"))
```

```
aux1 <- 2 * p * qnorm(1 - alpha / 2)^2 + 1
aux2 <- 2 * p * qnorm(1 - alpha)^2 + 1
if (alternative == "two.sided") {
z <- qnorm(1 - alpha / 2)
LL <- ((n - 1)^0.5 * detS) / ((n - 1)^0.5 + sqrt(2 * p) * z)
if (n > aux1)
UL <- ((n - 1)^0.5 * detS) / ((n - 1)^0.5 - sqrt(2 * p) * z) else
UL <- Inf
} else
if (alternative == "less") {
z <- qnorm(1 - alpha)
LL <- 0
if (n > aux2)
UL <- (n - 1)^0.5 * detS / ((n - 1)^0.5 - sqrt(2 * p) * z) else
UL <- Inf
} else
if (alternative == "greater") {
z <- qnorm(1 - alpha)
LL <- (n - 1)^0.5 * detS / ((n - 1)^0.5 + sqrt(2 * p) * z)
UL <- Inf
}
return(list(Zc = Zc, p.value = p.value, LL = LL, UL = UL))
}



# Sarkar approximation test for the normal generalized variance - ST
# H_0: |Sig| = Delta_0 and approximated confidence interval
sarkarGV.Test <- function(Delta_0, n, p, detS, alpha = 0.05,
alternative="two.sided")
{
j <- 1:p
muy <- sum(digamma((n - j) / 2)) + p * log(2)
sigy <- sqrt(sum(trigamma((n - j)/2)))
Zc <- (p * log(n - 1) + log(detS) - log(Delta_0) - muy) / sigy
if (alternative == "two.sided") p.value <- 2 * (1 - pnorm(abs(Zc)))
else
p.value <- pnorm(Zc, lower.tail = (alternative == "less"))
if (alternative == "two.sided") {
z <- qnorm(1 - alpha / 2)
LL <- exp(p * log(n - 1) + log(detS) - muy - sigy * z)
UL <- exp(p * log(n - 1) + log(detS) - muy + sigy * z)
} else
if (alternative == "less") {
z <- qnorm(1 - alpha)
LL <- 0
UL <- exp(p * log(n - 1) + log(detS) - muy + sigy * z)
} else
if (alternative == "greater") {
```

```
z <- qnorm(1 - alpha)
LL <- exp(p * log(n - 1) + log(detS) - muy - sigy * z)
UL <- Inf
}
return(list(Zc = Zc, p.value = p.value, LL = LL, UL = UL))
}


# DT
# Djauhari approximation test for the normal generalized variance
# H_0: |Sig| = Delta_0 and approximated confidence interval
djauhariGV.Test <- function(Delta_0, n, p, detS, alpha = 0.05,
alternative="two.sided")
{
j <- 1:p
b1 <- exp(sum(log(n - j)) - p * log(n - 1))
rb2 <- sqrt(exp(sum(log(n - j + 2)) -
          p * log(n - 1) + log(b1)) - b1^2)
Zc <- (detS / Delta_0 - b1) / rb2
aux1 <-  qnorm(1 - alpha / 2)^2 * rb2^2
aux2 <- qnorm(1 - alpha)^2 * rb2^2
if (alternative == "two.sided") p.value <- 2 * (1 - pnorm(abs(Zc)))
 else
p.value <- pnorm(Zc, lower.tail = (alternative == "less"))
if (alternative == "two.sided") {
z <- qnorm(1 - alpha / 2)
LL <- detS / (b1 + rb2 * z)
if (b1^2 > aux1 ) UL <- detS / (b1 - rb2 * z) else
UL <- Inf
} else
if (alternative == "less") {
z <- qnorm(1 - alpha)
LL <- 0
if (b1^2 > aux2) UL <- detS / (b1 - rb2 * z) else
UL <- Inf
} else
if (alternative == "greater") {
z <- qnorm(1 - alpha)
LL <- detS / (b1 + rb2 * z)
UL <- Inf
}
return(list(Zc = Zc, p.value = p.value, LL = LL, UL = UL))
}


# LRT for the normal generalized variance - LRT
# H_0: |Sig| = Delta_0 and approximated confidence interval
LRT.Test <- function(Delta_0, n, p, detS, alternative="two.sided")
```

```
{
detSigHat <- detS * ((n - 1) / n)^p
nu <- n  - 1
chi2c <- n * (log(Delta_0) - log(detSigHat)) +  n * p *
((detSigHat / Delta_0)^(1/p) - 1)
if (alternative == "two.sided") p.value <- 1 - pchisq(chi2c, 1) else
p.value <- pchisq(chi2c, 1, lower.tail = FALSE)
return(list(chi2c = chi2c, p.value = p.value))
}
```

```
BCLRT4.Test <- function(x, Delta_0, alternative="two.sided") - BCLRT
{
n <-     nrow(x)
p <-     ncol(x)
detS <- det(var(x))
detSigHat <- detS  * ((n - 1) / n)^p
nu <- n  - 1
EdetW  <- detSigHat   * prod(nu:(nu-p+1))
EdetW2 <- detSigHat^2 * prod(nu:(nu-p+1)) * prod((nu+2):(nu-p+3))
EdetW3 <- detSigHat^3 * prod(2^3 *
gamma((nu:(nu-p+1))/2 + 3) / gamma((nu:(nu-p+1))/2))
EdetW4 <- detSigHat^4 * prod(2^4 *
 gamma((nu:(nu-p+1))/2 + 4) / gamma((nu:(nu-p+1))/2))
mc2    <- EdetW2 - EdetW^2
mc3    <- 1/(nu^(3*p)) * (EdetW3 - EdetW^3 - 3*EdetW * mc2)
mc4    <- 1/(nu^(4*p)) * (EdetW4 - 5*EdetW^4 - 4* EdetW * EdetW3 +
 6 * EdetW^2 * EdetW2)
EdetS  <- EdetW/(nu^p)
ElogDetS <- -p*log(nu) + log(EdetW) -0.5*(EdetW2/EdetW^2 - 1) +
1/3 * mc3/EdetS^3 - 0.25*mc4/EdetS^4
EDetS1p <- 1/nu*(EdetW^(1/p) + 0.5*prod(1/p - 0:1) *
 EdetW^(1/p - 2) * mc2) +
1/(6*nu^(1 - 3*p)) * (prod(1/p - 0:2) * EdetW^(1/p - 3) * mc3) +
1/(24*nu^(1 - 4*p)) * (prod(1/p - 0:3) * EdetW^(1/p - 4) * mc4)
Eminus2logLamb <- n*log(Delta_0) - n*p - n*p*log((n-1)/n) -
n*ElogDetS + n*p/Delta_0^(1/p) * (n - 1) / n * EDetS1p
phi     <- nu / Eminus2logLamb
chi2c <- phi * (n * (log(Delta_0) - log(detSigHat)) +  n * p *
((detSigHat / Delta_0)^(1/p) - 1))
if (alternative == "two.sided") p.value <- 1 - pchisq(chi2c, 1) else
p.value <- pchisq(chi2c, 1, lower.tail = FALSE)
return(list(chi2c = chi2c, p.value = p.value))
}
```

```
# Monte Carlo Simulation Function to evaluate the
# test performance. Dependence: MASS
library(MASS)
evalMC <- function(N = 10000, n = 15, p = 2, eta = 0.2, m = 10000)
{
Rej <- matrix(0, 101, 18)
colnames(Rej) <- c("MCET10","AT10","ST10","DT10","LRT10","BCLRT10",
"MCET5","AT5","ST5","DT5","LRT5","BCLRT5",
"MCET1","AT1","ST1","DT1","LRT1","BCLRT1")

rDetSig <- seq(0.01, 5.00, by = 0.05)
rDetSig <- c(rDetSig[rDetSig<eta],eta, rDetSig[rDetSig > eta])
Rej  <- cbind(rDetSig, Rej)
mu <- rep(c(0), times  = p)
alternative <- "two.sided"
st <- 1.0 / N
ct <- 1
alpha <- 0.05
for (D in rDetSig)
{
#print(D)
Sigma <-  D^(1/p) * diag(p)
#print(det(Sigma))
rej10 <- rep(0.0, times = 6)
rej05 <- rep(0.0, times = 6)
rej01 <- rep(0.0, times = 6)
for (i in 1:N)
{
X <- mvrnorm(n, mu, Sigma)
detS <- det(cov(X))
MCET <- exactGV.Test(eta, n, p, detS, alpha, alternative, m)
AT <- andersonGV.Test(eta, n, p, detS, alpha, alternative)
ST <- sarkarGV.Test(eta, n, p, detS, alpha, alternative)
DT <- djauhariGV.Test(eta, n, p, detS, alpha, alternative)
LRT <- LRT.Test(eta, n, p, detS, alternative)
BCLRT <- BCLRT4.Test(X, eta, alternative = "two.sided")
if (MCET$p.value <= 0.10)    rej10[1] <- rej10[1] + st
if (MCET$p.value <= 0.05)    rej05[1] <- rej05[1] + st
if (MCET$p.value <= 0.01)    rej01[1] <- rej01[1] + st
if (AT$p.value <= 0.10)      rej10[2] <- rej10[2] + st
if (AT$p.value <= 0.05)      rej05[2] <- rej05[2] + st
if (AT$p.value <= 0.01)      rej01[2] <- rej01[2] + st
if (ST$p.value <= 0.10)      rej10[3] <- rej10[3] + st
if (ST$p.value <= 0.05)      rej05[3] <- rej05[3] + st
if (ST$p.value <= 0.01)      rej01[3] <- rej01[3] + st
if (DT$p.value <= 0.10)      rej10[4] <- rej10[4] + st
if (DT$p.value <= 0.05)      rej05[4] <- rej05[4] + st
if (DT$p.value <= 0.01)      rej01[4] <- rej01[4] + st
```

```
if (LRT$p.value <= 0.10)      rej10[5] <- rej10[5] + st
if (LRT$p.value <= 0.05)      rej05[5] <- rej05[5] + st
if (LRT$p.value <= 0.01)      rej01[5] <- rej01[5] + st
if (BCLRT$p.value <= 0.10)  rej10[6] <- rej10[6] + st
if (BCLRT$p.value <= 0.05)  rej05[6] <- rej05[6] + st
if (BCLRT$p.value <= 0.01)  rej01[6] <- rej01[6] + st
}
rej <- c(rej10, rej05, rej01)
Rej[ct,2:19] <- rej
ct <- ct + 1
}
return(Rej)
}
```

Graphics

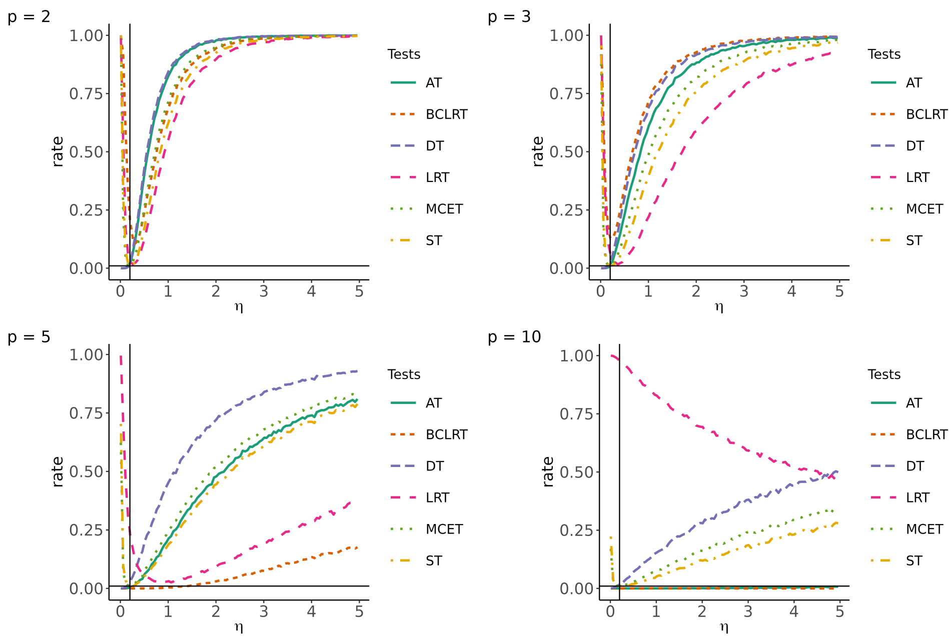

Figure 7. Performance evaluation of hypothesis tests at the 1% significance level for $\eta = 0.2$, $n = 15$ and $p = 2, 3, 5$ and 10.

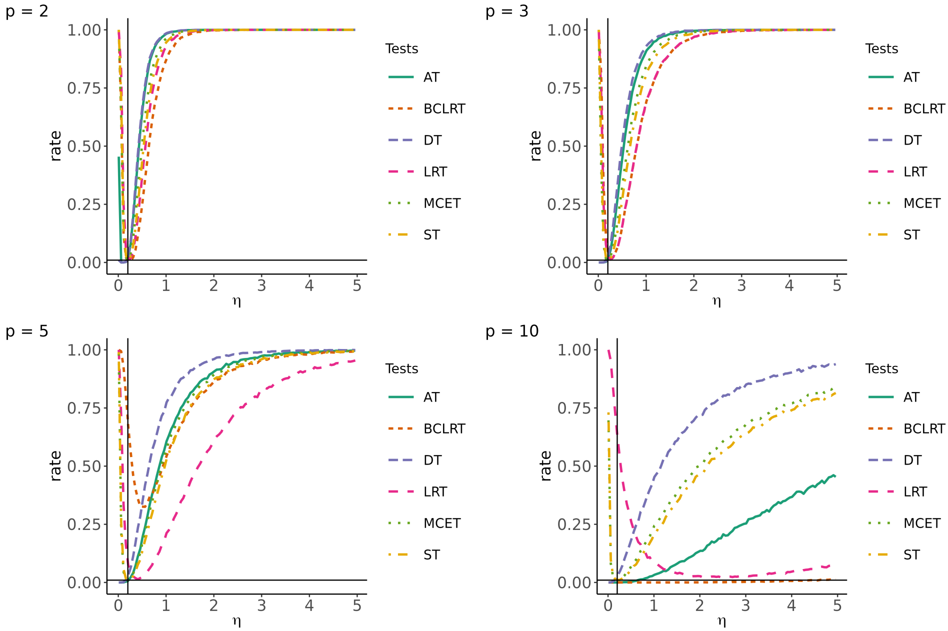

Figure 8. Performance evaluation of hypothesis tests at the 1% significance level for $\eta = 0.2$, $n = 30$ and $p = 2, 3, 5$ and 10.

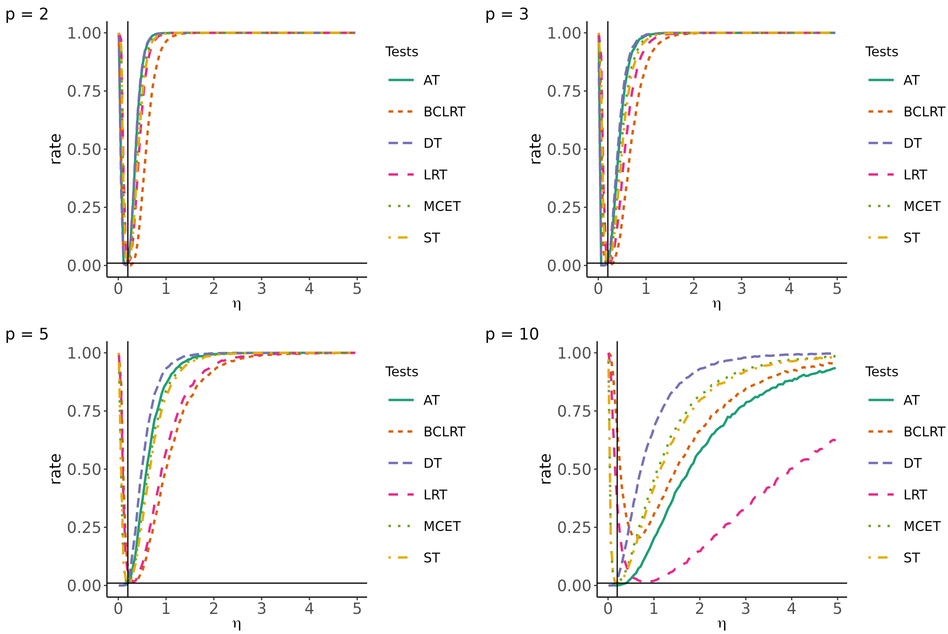

Figure 9. Performance evaluation of hypothesis tests at the 1% significance level for $\eta = 0.2$, $n = 50$ and $p = 2, 3, 5$ and 10.

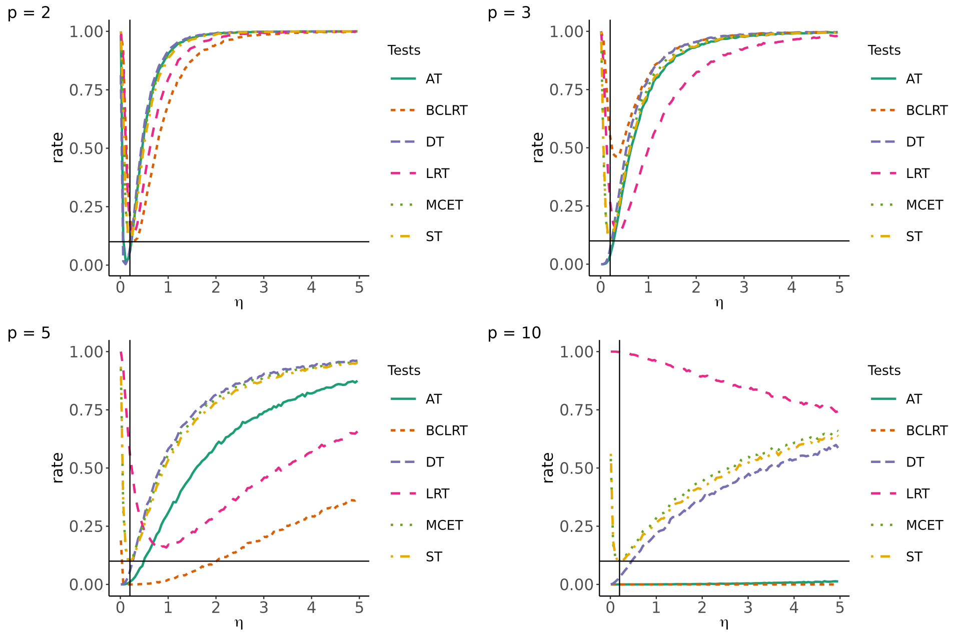

Figure 10. Performance evaluation of hypothesis tests at the 10% significance level for $\eta = 0.2$, $n = 15$ and $p = 2, 3, 5$ and 10.

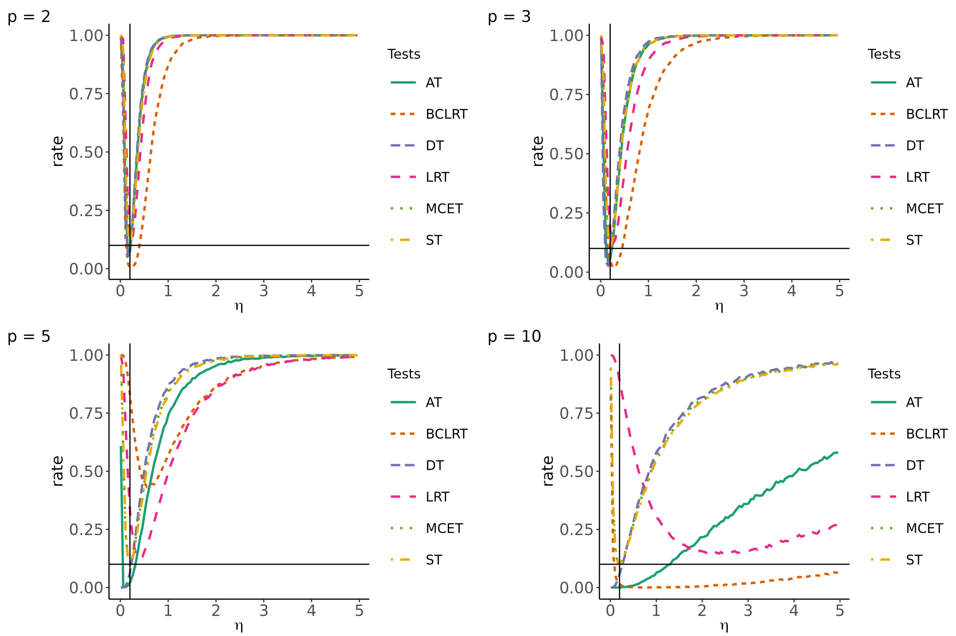

Figure 11. Performance evaluation of hypothesis tests at the 10% significance level for $n = 30$ and $p = 2, 3, 5$ and 10.

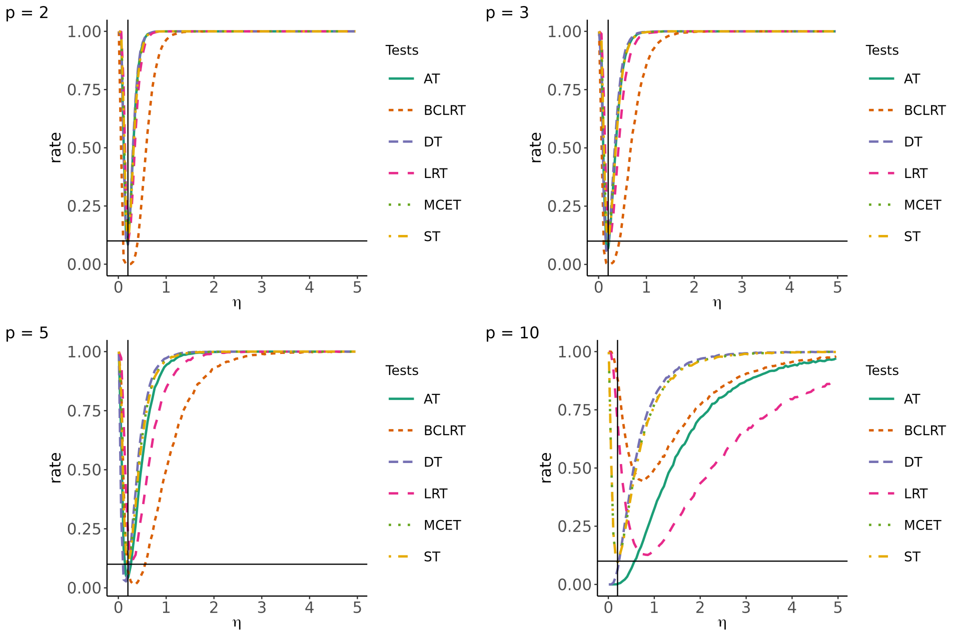

Figure 12. Performance evaluation of hypothesis tests at the 10% significance level for $\eta = 0.2$, $n = 50$ and $p = 2, 3, 5$ and 10.

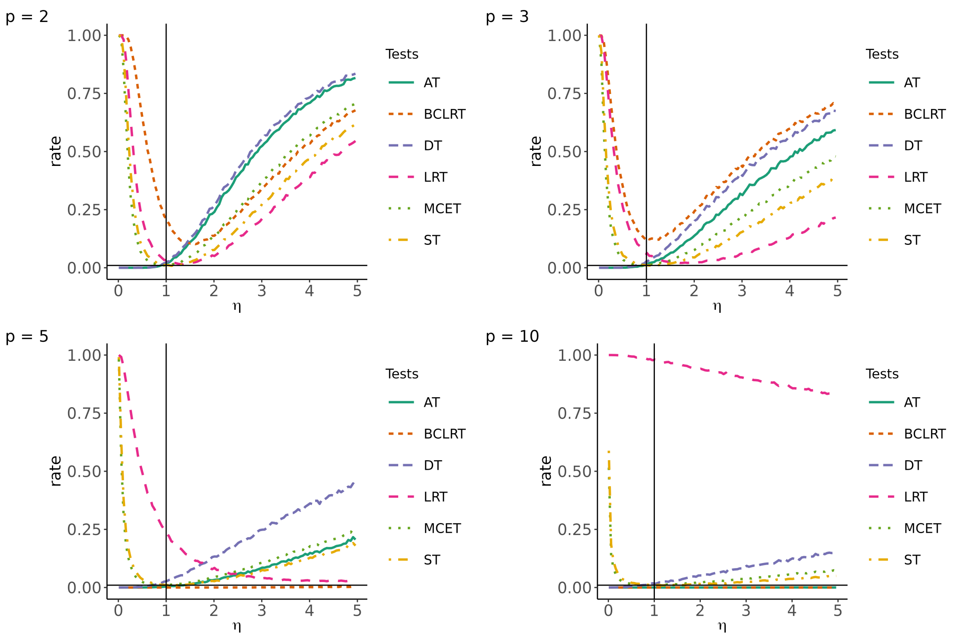

Figure 13. Performance evaluation of hypothesis tests at the 1% significance level for $\eta = 1$, $n = 15$ and $p = 2, 3, 5$ and 10.

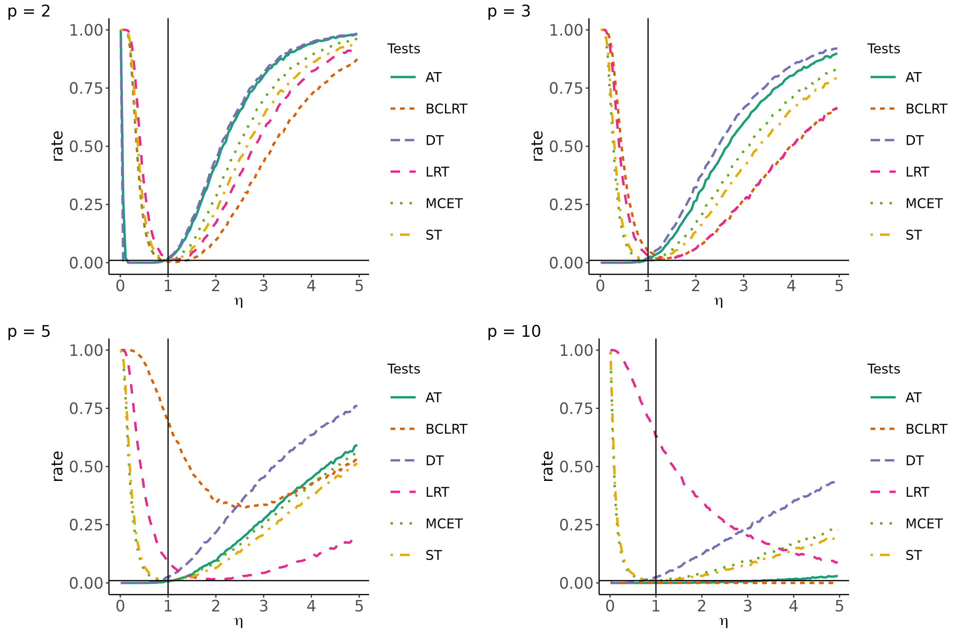

Figure 14. Performance evaluation of hypothesis tests at the 1% significance level for $\eta = 1$, $n = 30$ and $p = 2, 3, 5$ and 10.

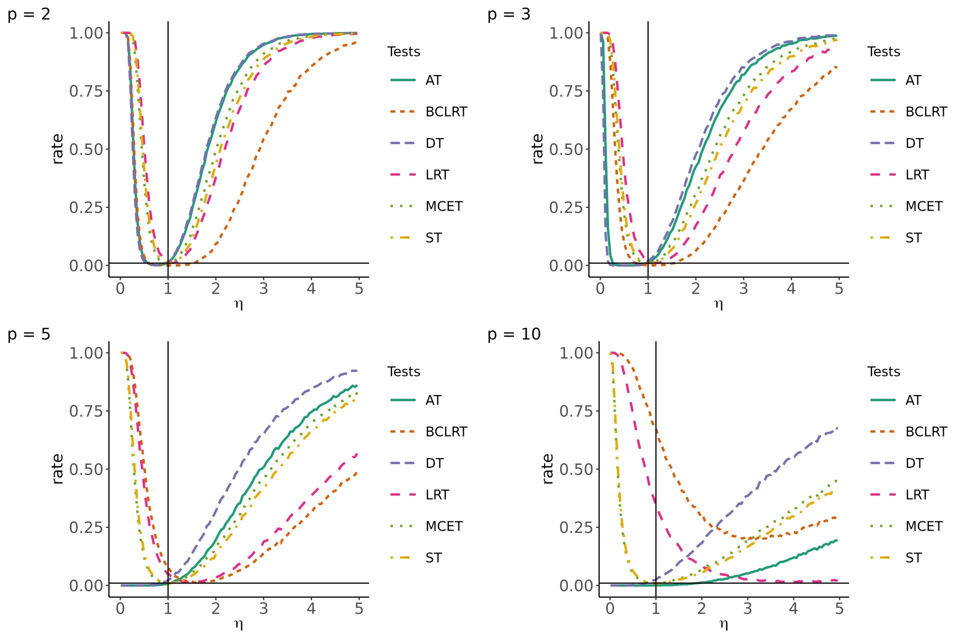

Figure 15. Performance evaluation of hypothesis tests at the 1% significance level for $\eta = 1$, $n = 50$ and $p = 2, 3, 5$ and 10.

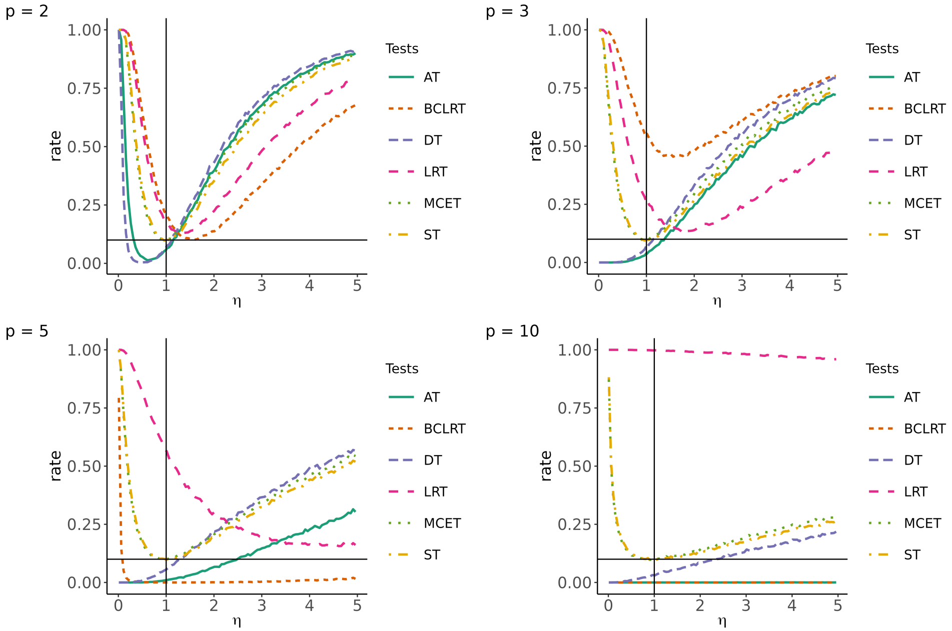

Figure 16. Performance evaluation of hypothesis tests at the 10% significance level for $\eta = 1$, $n = 15$ and $p = 2, 3, 5$ and 10.

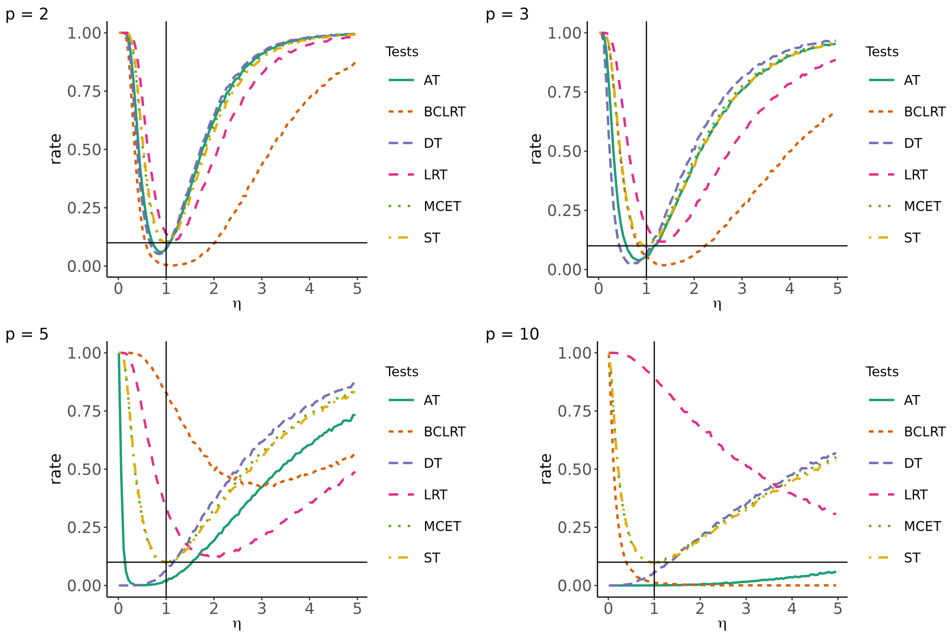

Figure 17. Performance evaluation of hypothesis tests at the 10% significance level for $\eta = 1$, $n = 30$ and $p = 2, 3, 5$ and 10.

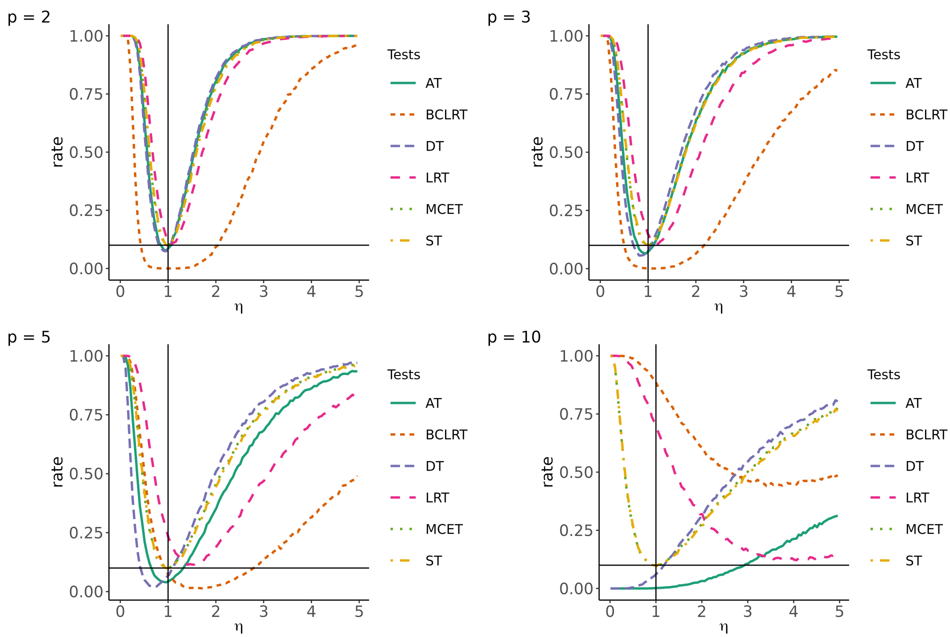

Figure 18. Performance evaluation of hypothesis tests at the 10% significance level for $\eta = 1$, $n = 50$ and $p = 2, 3, 5$ and 10.