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ARTICLE

The Poisson-Rama distribution with properties and applications to model over-dispersed count data[1]

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Abstract
The discrete data available in any fields of knowledge is influenced by several known and unknown factors and the factors which affect the discrete data are stochastic in nature. The stochastic nature of discrete data is really a challenge for statistician to model and analyse with the existing discrete distributions. In the present paper, Poisson-Rama distribution, a Poisson mixture of Rama distribution, has been proposed to model over-dispersed data. Distributional properties, estimation of parameter using maximum likelihood method, and applications of the proposed distributions have been discussed. The simulation study has been carried out to know the consistency of maximum likelihood estimates of parameter. It is observed that the proposed distribution gives much closure fit than several over-dispersed one parameter discrete distributions including Poisson-Lindley distribution, Poisson-Akash distribution and Poisson-Ishita distribution.
Keywords: Count Data; Rama distribution; Compounding; Statistical Properties; Estimation; Goodness of fit.

1. Introduction

The Poisson distribution is a suitable distribution for equi-dispersed (mean equal to variance) count data. Count data appear in several fields of knowledge including biological sciences, insurance, medicine and agriculture, some among others. But in real life situation, it has been observed that most of the datasets being stochastic in nature are either over-dispersed (variance greater than mean) or under-dispersed (variance less than mean). Various statistical techniques are proposed to deal with over-dispersed count data such as weighted distributions and the mixture of distributions. A well-known and widely used technique for allowing over-dispersion in count data is the mixed Poisson distribution.

During recent decades an attempt has been made by different researchers to derive over-dispersed one parameter discrete distribution by compounding Poisson distribution with one parameter continuous lifetime distributions. One of the important characteristics of the Poisson mixture of lifetime distribution is that the resultant distribution follows some characteristics of its mixing distribution. A popular one parameter over-dispersed discrete distribution is the Poisson-Lindley distribution (PLD) proposed by Sankaran (1970). The PLD is a Poisson mixture of Lindley distribution introduced by Lindley (1958).

The second popular one parameter over-dispersed discrete distribution is the Poisson-Akash distribution (PAD) proposed by Shanker (2017). The PAD is a Poisson mixture of Akash distribution introduced by Shanker (2015). The third popular over dispersed discrete distribution is the Poisson-Ishita distribution (PID) proposed by Shukla & Shanker (2019). The PID is a Poisson-mixture of Ishita distribution introduced by Shanker & Shukla (2017). Further, it has been observed that this one parameter discrete distributions are not suitable for some over-dispersed datasets from biological sciences due to their levels of over-dispersion. Shanker & Hagos (2015) have detailed discussion on applications of PLD for data arising from biological sciences, as the data from biological sciences are, in general, over-dispersed. It has been observed by Shanker & Hagos (2015) that in some biological science data PLD does not give better fit and hence there is a need for another over-dispersed discrete distribution.

Shanker (2017) introduced a one parameter lifetime distribution, named Rama distribution, defined by its probability density function (pdf) and cumulative distribution function (cdf)

$$f(x;\theta)=\frac{\theta^4}{\theta^3+6}\left(1+x^3\right)e^{-\theta x};x>0,\theta>0 \tag{1}$$

$$F(x;\theta)=1-\left[1+\frac{\theta x\left(\theta^2x^2+3\theta x+6\right)}{\theta^3+6}\right]e^{-\theta x};x>0,\theta>0 \tag{2}$$

Statistical properties including moments and moments based measures, hazard rate function, mean residual life function, stochastic ordering and deviations from the mean and the median, estimation of parameter and applications are available in Shanker (2017).

The main purpose of this paper is to derive an over-dispersed discrete distribution which is the compound of the Poisson and the Rama distribution. Descriptive statistical constants including coefficients of variation, skewness, kurtosis and index of dispersion have been studied. Over-dispersion, unimodality and increasing hazard rate of the derived distribution has been discussed. Maximum likelihood method and the method of moment have been explained to estimate parameter of the proposed distribution. Goodness of fit and its comparison with other one parameter over-dispersed discrete distributions are presented.

2. The Poisson-Rama Distribution

The one of the important characteristics of the Poisson distribution is that although the Poisson distribution is equi-dispersed discrete distribution but the compounds of Poisson distribution with any lifetime distribution will always results into an over-dispersed discrete distribution.

Let X follows Poisson distribution with parameter $\lambda>0$ having probability mass function (pmf)

$$P(X\mid\lambda)=\frac{e^{-\lambda}\lambda^x}{x!};x=0,1,2,...$$

Now suppose the parameter λ follows Rama distribution with parameter θ having pdf

$$f(\lambda\mid\theta)=\frac{\theta^4}{\theta^3+6}\left(1+\lambda^3\right)e^{-\theta\lambda};\lambda>0,\theta>0$$

Thus, the marginal pmf of X can be obtained as

$$P\left(X=x\right)=\int_0^\infty P\left(X\,|\,\lambda\right)f\left(\lambda\,|\,\theta\right)d\lambda=\int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\frac{\theta^4}{\theta^3+6}\left(1+\lambda^3\right)e^{-\theta\lambda}d\lambda \tag{3}$$

$$=\frac{\theta^4}{\left(\theta^3+6\right)x!}\int_0^\infty e^{-(\theta+1)\lambda}\left(\lambda^x+\lambda^{x+3}\right)d\lambda$$

$$=\frac{\theta^4}{\theta^3+6}\frac{x^3+6x^2+11x+\left(\theta^3+3\theta^2+3\theta+7\right)}{\left(\theta+1\right)^{x+4}};x=0,1,2,...,\theta>0 \tag{4}$$

We would call this distribution as Poisson-Rama distribution (PRD). Further, we would show that the pmf of PRD is over-dispersed, unimodal and has increasing hazard rate. The behavior of the pmf of PRD for varying values of parameter has been shown in the following figure1 and it reveals that as the value of parameter increases, the distribution becomes positively skewed to the right and over-dispersed. From the figure 1, it is quite obvious that the PRD has a tendency to accommodate right tail and for particular values of the parameter, the tail tends to zero at a much faster rate, which means that the PRD fits appropriately to those datasets where there is an extended right tail or the tail approaches to zero at a faster rate. The field of biology and the insurance are flooded with such over-dispersed datasets where the right tail approaches to zero at a very faster rate.

Further, it should be noted that the PRD is actually a two-component mixture distribution that can be expressed as

$$P\left(x;\theta\right)=p\,P_1\left(x;\theta\right)+\left(1-p\right)P_2\left(x;\theta\right),$$

where $P_i\left(x;\theta\right)$ is the pmf of the negative binomial distribution with parameter the number of successes i and proportion $\frac{\theta}{\theta+1}$. When $i=1$, $P_i\left(x;\theta\right)$ is the pmf of the geometric distribution, which is a special case of the negative binomial distribution. The formulae for p and $P_i\left(x;\theta\right)$ for $i=1,2$ are given by

$$p=\frac{\theta^3}{\theta^3+6},P_1\left(x;\theta\right)=\frac{\theta}{\left(\theta+1\right)^{x+1}},P_2\left(x;\theta\right)=\frac{\left(x+1\right)\left(x+2\right)\left(x+3\right)\theta^4}{6\left(\theta+1\right)^{x+4}}$$

Now, even though the PRD is two-component mixtures of negative binomial distribution, the presence of two modes is not visible in any of the plots of pmf of PRD in figure 1 for the selected values of the parameter θ. This suggests that the two modes which come from the two sub-populations must be located very close to each other. As observed by Tajuddin *et al* (2022) that if the modes of the sub-populations are located very close to each other, the population will have single mode. This means that if the existence of the modes of the sub-populations each with very close modes values are certain, then this distribution can be considered as one of the candidates for model fitting of over-dispersed data.

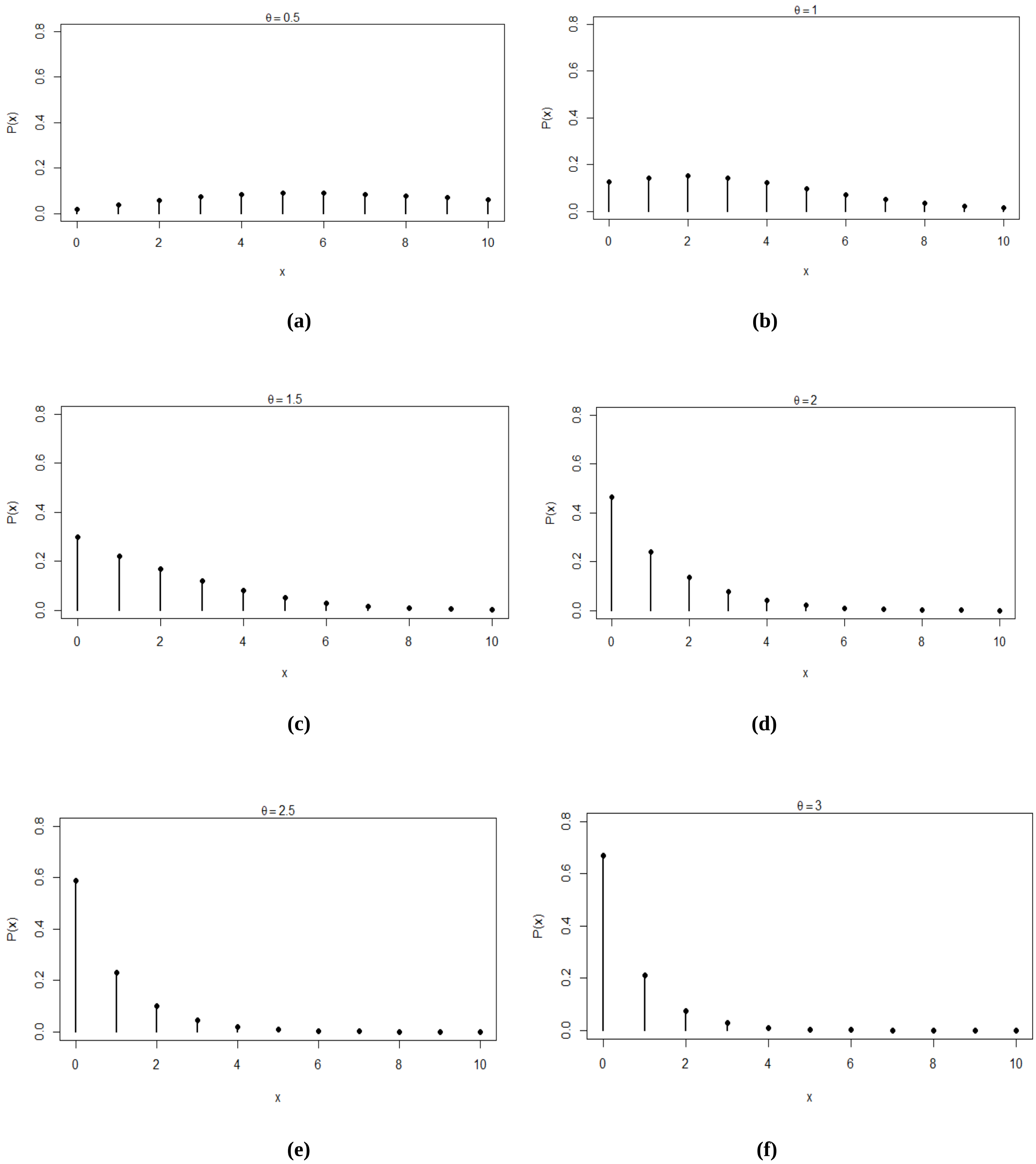

Figure 1. Probability mass function of Poisson-Rama distribution for varying values of parameter.

3. Descriptive Statistical Constants

Using (2.1), the r th factorial moment about origin, $\mu_{(r)}'$, of PRD can be obtained as

$$\mu_{(r)}' = E\left[E\left(X^{(r)} \mid \lambda\right)\right] = \frac{\theta^4}{\theta^3+6}\int_0^\infty \left[\sum_{x=0}^{\infty} x^{(r)} \frac{e^{-\lambda}\lambda^x}{x!}\right]\left(1+\lambda^3\right)e^{-\theta\lambda}d\lambda \tag{5}$$

$$= \frac{\theta^4}{\theta^3+6}\int_0^\infty \lambda^r \left[\sum_{x=r}^{\infty} \frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right]\left(1+\lambda^3\right)e^{-\theta\lambda}d\lambda$$

$$= \frac{\theta^4}{\theta^3+6}\int_0^\infty \lambda^r \left(1+\lambda^3\right)e^{-\theta\lambda}d\lambda$$

$$= \frac{r!\left\{\theta^3+(r+1)(r+2)(r+3)\right\}}{\theta^r\left(\theta^3+6\right)}; r=1,2,3,\ldots. \tag{6}$$

Substituting $r=1,2,3$ and 4, the first four factorial moment about origin of PRD can be obtained as

$$\mu_{(1)}' = \frac{\theta^3+24}{\theta\left(\theta^3+6\right)} \tag{7}$$

$$\mu_{(2)}' = \frac{2\left(\theta^3+60\right)}{\theta^2\left(\theta^3+6\right)} \tag{8}$$

$$\mu_{(3)}' = \frac{6\left(\theta^3+120\right)}{\theta^3\left(\theta^3+6\right)} \tag{9}$$

$$\mu_{(4)}' = \frac{24\left(\theta^3+210\right)}{\theta^4\left(\theta^3+6\right)} \tag{10}$$

The relationship between moments about origin and factorial moments about origin gives the following four moments about origin

$$\mu_1' = \frac{\theta^3+24}{\theta\left(\theta^3+6\right)} \tag{11}$$

$$\mu_2' = \frac{\theta^4+2\theta^3+24\theta+120}{\theta^2\left(\theta^3+6\right)} \tag{12}$$

$$\mu_3' = \frac{\theta^5+6\theta^4+6\theta^3+24\theta^2+360\theta+720}{\theta^3\left(\theta^3+6\right)} \tag{13}$$

$$\mu_4' = \frac{\theta^6+14\theta^5+36\theta^4+48\theta^3+840\theta^2+4230\theta+5040}{\theta^4\left(\theta^3+6\right)} \tag{14}$$

Using the relationship between moments about mean and the moments about origin, moments about the mean of PRD are obtained as

$$\mu_2 = \frac{\theta^7 + \theta^6 + 30\theta^4 + 84\theta^3 + 144\theta + 144}{\theta^2\left(\theta^3+6\right)^2} \tag{15}$$

$$\mu_3 = \frac{\begin{pmatrix}\theta^{11} + 3\theta^{10} + 2\theta^9 + 36\theta^8 + 270\theta^7 + 396\theta^6 + 324\theta^5 + 1944\theta^4 + 648\theta^3 \\ +864\theta^2 + 2592\theta + 1728\end{pmatrix}}{\theta^3\left(\theta^3+6\right)^3} \tag{16}$$

$$\mu_4 = \frac{\begin{pmatrix}\theta^{15} + 10\theta^{14} + 18\theta^{13} + 51\theta^{12} + 852\theta^{11} + 3132\theta^{10} + 3348\theta^9 + 11880\theta^8 \\ +349992\theta^7 + 235444\theta^6 + 59184\theta^5 + 132192\theta^4 + 98496\theta^3 + 98496\theta^2 \\ +186624\theta + 93312\end{pmatrix}}{\theta^4\left(\theta^3+6\right)^4} \tag{17}$$

Now, the descriptive measures of PRD including coefficient of variation (C.V), skewness, kurtosis and index of dispersion are obtained as

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\theta^7 + \theta^6 + 30\theta^4 + 84\theta^3 + 144\theta + 144}}{\theta^3 + 24} \tag{18}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\left(\mu_2\right)^{3/2}} = \frac{\begin{pmatrix}\theta^{11} + 3\theta^{10} + 2\theta^9 + 36\theta^8 + 270\theta^7 + 396\theta^6 + 324\theta^5 + 1944\theta^4 + 648\theta^3 \\ +864\theta^2 + 2592\theta + 1728\end{pmatrix}}{\left(\theta^7 + \theta^6 + 30\theta^4 + 84\theta^3 + 144\theta + 144\right)^{3/2}} \tag{19}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^{\,2}} = \frac{\begin{pmatrix}\theta^{15} + 10\theta^{14} + 18\theta^{13} + 51\theta^{12} + 852\theta^{11} + 3132\theta^{10} + 3348\theta^9 + 11880\theta^8 \\ +349992\theta^7 + 235444\theta^6 + 59184\theta^5 + 132192\theta^4 + 98496\theta^3 + 98496\theta^2 \\ +186624\theta + 93312\end{pmatrix}}{\left(\theta^7 + \theta^6 + 30\theta^4 + 84\theta^3 + 144\theta + 144\right)^2} \tag{20}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{\theta^7 + \theta^6 + 30\theta^4 + 84\theta^3 + 144\theta + 144}{\theta\left(\theta^3+6\right)\left(\theta^3+24\right)} \tag{21}$$

The behaviors of coefficients of variation, skewness, kurtosis and index of dispersion of PRD for varying values of parameter are shown in the following figure 2. It is obvious that the coefficient of variation, skewness, kurtosis are all increasing for increasing values of parameter while the index of dispersion is decreasing for increasing values of parameter.

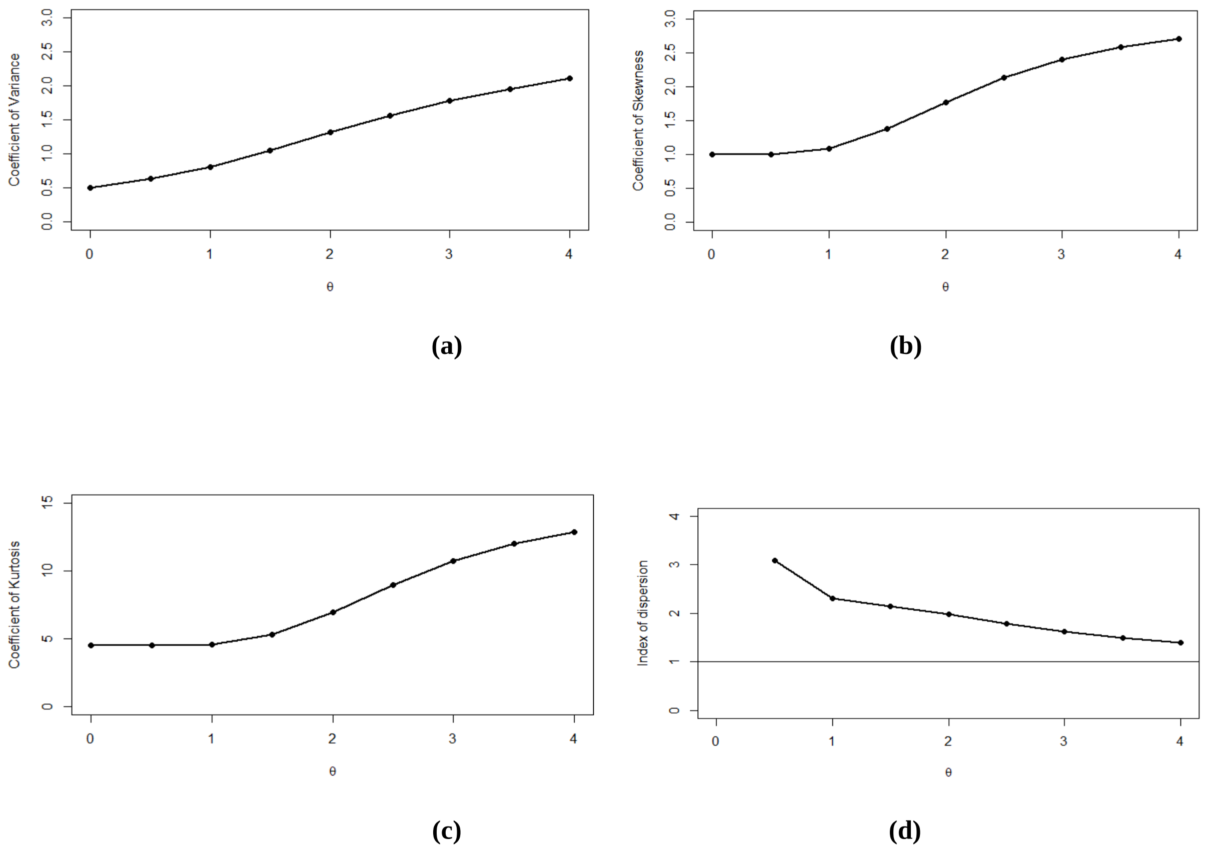

Figure 2. Coefficients of variation, skewness, kurtosis and index of dispersion for varying values of parameter.

4. Statistical Properties

4.1 Over-dispersion

We have

$$\mu_2 = \frac{\theta^7 + \theta^6 + 30\theta^4 + 84\theta^3 + 144\theta + 144}{\theta^2\left(\theta^3+6\right)^2}$$

$$= \frac{\theta^3+24}{\theta\left(\theta^3+6\right)}\left[\frac{\theta^7 + \theta^6 + 30\theta^4 + 84\theta^3 + 144\theta + 144}{\theta\left(\theta^3+6\right)\left(\theta^3+24\right)}\right]$$

$$= \frac{\theta^3+24}{\theta\left(\theta^3+6\right)}\left[1+\frac{\theta^6 + 84\theta^3 + 144}{\theta\left(\theta^3+6\right)\left(\theta^3+24\right)}\right]$$

$$= \mu_1'\left[1+\frac{\theta^6 + 84\theta^3 + 144}{\theta\left(\theta^3+6\right)\left(\theta^3+24\right)}\right] \qquad (22)$$

This shows that $\mu_2 > \mu_1'$ and thus PRD is always over-dispersed distribution. Therefore, PRD can be used for discrete data sets which are over-dispersed in nature.

4.2 Increasing hazard rate and unimodality

The PRD distribution is skewed to the right, unimodal and decreasing which is supported by the following result. It can be easily shown that PRD has increasing hazard rate (IHR) and is unimodal. Since

$$F(x,\theta) = \frac{P(x+1,\theta)}{P(x,\theta)} = \frac{1}{\theta+1}\left[1 + \frac{3x^2+15x+18}{x^3+6x^2+11x+\left(\theta^3+3\theta^2+3\theta+7\right)}\right]$$

is a decreasing function of x for a given $\theta, P(x,\theta)$ is log-concave. This implies that PRD has an increasing hazard rate and is unimodal. Grandell (1997) has detailed discussion about relationship between log-concavity, IHR and Unimodality of discrete distributions.

Theorem 1: $F(x,\theta)$ is a decreasing function of x for a given of θ.

Proof: We have,

$$F(x,\theta) = \frac{P(x+1,\theta)}{P(x,\theta)} = \frac{1}{\theta+1}\left[1 + \frac{3x^2+15x+18}{x^3+6x^2+11x+\left(\theta^3+3\theta^2+3\theta+7\right)}\right]$$

$$F'(x,\theta) = \frac{(6x+15)\left[x^3+6x^2+11x+\left(\theta^3+3\theta^2+3\theta+7\right)\right]-\left(3x^2+15x+18\right)\left(3x^2+12x+11\right)}{(\theta+1)\left[x^3+6x^2+11x+\left(\theta^3+3\theta^2+3\theta+7\right)\right]^2}$$

$$= \frac{-3x^4-30x^3-78x^2-174x-126+\left(\theta^2+3\theta+3\right)(6\theta x+15\theta)}{(\theta+1)\left[x^3+6x^2+11x+\left(\theta^3+3\theta^2+3\theta+7\right)\right]^2} < 0$$

Since $F'(x,\theta) < 0$, thus $F(x,\theta)$ is a decreasing function of x for a given of θ

Theorem 2: The pmf $P(x,\theta)$ of PRD is log-concave.

Proof: we have

$$P(x,\theta) = \frac{\theta^4}{\theta^3+6}\,\frac{x^3+6x^2+11x+\left(\theta^3+3\theta^2+3\theta+7\right)}{(\theta+1)^{x+4}}$$

$$\log\left(P(x,\theta)\right) = \log\left(\frac{\theta^4\left[x^3+6x^2+11x+\left(\theta^3+3\theta^2+3\theta+7\right)\right]}{\left(\theta^3+6\right)(\theta+1)^{x+4}}\right)$$

$$\log\left(P(x,\theta)\right)=4\log(\theta)+\log\left[x^3+6x^2+11x+\left(\theta^3+3\theta^2+3\theta+7\right)\right]-\log\left(\theta^3+6\right)-(x+4)\log(\theta+1)$$

Assuming $g(x)=\log\left(P(x,\theta)\right)$, we have

$$g'(x)=\frac{3x^2+12x+11}{x^3+6x^2+11x+\left(\theta^3+3\theta^2+3\theta+7\right)}-\log(\theta+1) \text{ and}$$

$$g''(x)=\frac{(6x+12)\left(x^3+6x^2+11x+\left(\theta^3+3\theta^2+3\theta+7\right)\right)-\left(3x^2+12x+11\right)^2}{\left[x^3+6x^2+11x+\left(\theta^3+3\theta^2+3\theta+7\right)\right]^2}$$

$$=\frac{-3x^4-24x^3-72x^2-90x-37+6\left(\theta^3+3\theta^2+3\theta\right)(x+2)}{\left[x^3+6x^2+11x+\left(\theta^3+3\theta^2+3\theta+7\right)\right]^2}<0$$

Thus, the pmf $P(x,\theta)$ of PRD is log-concave

5. Estimation of Parameter

5.1 Method of Moment Estimate

Let $x_1, x_2, ..., x_n$ be a random sample of size n from PRD. Equating the first moment about origin to the corresponding sample moment, the moment estimate (ME) $\tilde{\theta}$ of θ is the solution of the following fourth degree polynomial equation $\bar{x}\theta^4-\theta^3+6\theta\bar{x}-24=0$, where $\bar{x}$ is the sample mean.
This equation can be solved using Newton-Raphson method to get the estimate of the parameter. The initial value of θ in the Newtin-Raphson method for the above polynomial equation can be selected as follow: Suppose $f(\theta)=\bar{x}\theta^4-\theta^3+6\theta\bar{x}-24$, where $\bar{x}$ is known from the dataset for which we are estimating the value of the parameter. Now we have to guess two values of θ, say θ_1 and θ_2 such that $f(\theta_1)f(\theta_2)<0$. Then we can select any value of θ say θ_0 between θ_1 and θ_2 as initial value value of θ in the Newton-Raphson method.

5.2 Maximum Likelihood Estimate

Let $x_1, x_2, ..., x_n$ be a random sample of size n from PRD and let f_x be the observed frequency in the sample corresponding to $X = x\ (x=1,2,3,...,k)$ such that $\sum_{x=1}^{k} f_x = n$, where k is the largest observed value having non-zero frequency. The likelihood function L of PRD is given by

$$L=\left(\frac{\theta^4}{\theta^3+6}\right)^n \frac{1}{(\theta+1)^{\sum_{x=1}^{k}(x+4)f_x}} \prod_{x=1}^{k}\left[x^3+6x^2+11x+\left(\theta^3+3\theta^2+3\theta+7\right)\right]^{f_x} \quad (23)$$

The log- likelihood function is obtained as

$$\log L = n\log\left(\frac{\theta^4}{\theta^3+6}\right)-\sum_{x=1}^{k} f_x\,(x+4)\log(\theta+1)+\sum_{x=1}^{k} f_x \log\left[x^3+6x^2+11x+\left(\theta^3+3\theta^2+3\theta+7\right)\right] \quad (24)$$

The first derivative of the log- likelihood function is given by

$$\frac{\partial \log L}{\partial\theta}=\frac{4n}{\theta}-\frac{3n\theta^2}{\theta^3+6}-\frac{n(\bar{x}+4)}{\theta+1}+\sum_{x=1}^{k}\frac{\left(3\theta^2+6\theta+3\right)f_x}{x^3+6x^2+11x+\left(\theta^3+3\theta^2+3\theta+7\right)}, \quad (25)$$

where $\bar{x}$ is the sample mean.

The maximum likelihood estimate (MLE), $\hat{\theta}$ of θ is the solution of the equation $\frac{\partial \log L}{\partial\theta}=0$ and is given by the solution of the non-linear equation

$$\frac{n\left(\theta^3+24\right)}{\theta\left(\theta^3+6\right)}-\frac{n(\bar{x}+4)}{\theta+1}+\sum_{x=1}^{k}\frac{\left(3\theta^2+6\theta+3\right)f_x}{x^3+6x^2+11x+\left(\theta^3+3\theta^2+3\theta+7\right)}=0 \quad (26)$$

Since this log-likelihood equation cannot be expressed in closed form, it may be difficult to solve it by direct method. Therefore, the MLE of the parameter θ can be computed iteratively by solving log-likelihood equation using Newton-Raphson iteration available in R-software, until sufficiently close values of the parameter θ is obtained. The initial value of the parameter θ can be taken as the value given by method of moment estimate.

6. A Simulation Study

In this section, the performance of the proposed distribution has been studied by a simulation technique. The simulation process consists in generating N=10,000 pseudo-random samples of sizes n=50, 100, 150, 200, 300, and 400 of a variable X having PSD (2.2). The procedure is based on the Monte Carlo simulation method to estimate the average bias and the mean squared error (MSE) of the MLEs of the parameter θ. The following formulae used for finding bias and MSE of parameter are.

$$B(\hat{\theta})=\frac{1}{N}\sum_{j=1}^{N}(\hat{\theta}_j-\theta), \qquad \mathrm{MSE}(\hat{\theta})=\frac{1}{N}\sum_{j=1}^{N}(\hat{\theta}_j-\theta)^2)$$

The following algorithm can be used to generate a single random variable from PSD

Algorithm:

Generate, $u \sim U(0,1)$

$x \to 0$

$p_x \Rightarrow \theta^4(\theta^3 + 3\theta^2 + 3\theta + 7)/(\theta + 1)^4(\theta^3 + 6)$

while$(p_x < u)$do

$x \to x + 1$

$p_{x1} = p_x * p_{x-1}$

$p_x \Rightarrow p_x + p_{x1}$

while

return(x)

end

Table 1. Estimated Bias and MSE of MLEs $(\hat{\theta})$

Sample Size	True Parameter Value of θ	Estimated Parameter value of θ	Bias	MSE
n=50	2.0	2.6876	0.0137	0.00090
	2.5	3.2949	0.0158	0.01182
	3.0	3.5937	0.0118	0.00664
	3.5	4.0029	0.0100	0.00483
n=100	2.0	2.9159	0.0091	0.00778
	2.5	3.3893	0.0088	0.00780
	3.0	3.8513	0.0085	0.00713
	3.5	4.2683	0.0076	0.00580
n=150	2.0	0.9374	0.0062	0.00539
	2.5	3.4920	0.0066	0.00645
	3.0	3.8045	0.0053	0.00410
	3.5	4.3012	0.0053	0.00394
n=200	2.0	2.9159	0.0045	0.00409
	2.5	3.3893	0.0044	0.00385
	3.0	3.7158	0.0035	0.00246
	3.5	4.2203	0.0036	0.00253
n=300	2.0	2.9374	0.0031	0.00266
	2.5	3.3255	0.0027	0.00220
	3.0	3.3255	0.0010	0.00030
	3.5	4.3086	0.0026	0.00161
n=400	2.0	2.9238	0.00230	0.00205
	2.5	3.4016	0.00225	0.00200
	3.0	3.8531	0.00213	0.00171
	3.5	4.3434	0.00210	0.00172

The Table 1 shows the bias and MSE of the MLEs of the parameter θ for different sample sizes. The result indicates that bias and mean square error tends to zero when the sample size increases, which confirms the asymptotic theory of maximum likelihood estimator.

7. Goodness of fit

In this section, three over-dispersed count data are analyzed to show the applications of PRD. The first dataset is the number of mistakes in copying groups of random digits, available in Kemp & Kemp (1965). This data is over-dispersed with the index of dispersion of 1.605. The second data is the Pyrausta nublilalis, available in Beall (1940) with an index of dispersion of 1.758.

The third data is the number of Chromatid aberrations. Studies on chromosomal aberrations and Chromatid aberrations have been performed over the past several years and the percentage of cells with chromosomal aberrations and Chromatid aberrations have been used as a quantitative measure of biological dosimetry. In the analysis of data on chemically induced chromosome aberrations in cultures of human leukocytes, Loeschke & Kohler (1976) and Janardan & Schaeffer (1977) recommended modified Poisson distribution, negative binomial distribution and Lagrangian Poisson distribution. But these distributions do not provide good fit because the chemically induced chromosome aberrations and Chromatid aberrations are time dependent and hence have higher level of over- dispersion than the over-dispersion of negative binomial distribution. The index of dispersion of the third data is of 2.05. The goodness of fit of PRD has been compared with the Poisson distribution and one parameter over-dispersed distributions namely Poisson-Lindley distribution (PLD), Poisson-Akash distribution (PAD) and Poisson-Ishita distribution (PID). The pmf of PLD, PAD and PID is given in the following table 2.

Table 2. Probability mass function of over-dispersed discrete probability distributions

Name of the distribution	Pmf
PLD	$P(x,\theta)=\dfrac{\theta^2(x+\theta+2)}{(\theta+1)^{x+3}}\ ;x=0,1,2,...,\theta>0$
PAD	$P(x;\theta)=\dfrac{\theta^3}{\theta^2+2}\cdot\dfrac{x^2+3x+(\theta^2+2\theta+3)}{(\theta+1)^{x+3}}\ ;x=0,1,2,...,\theta>0$
PID	$P(x;\theta)=\dfrac{\theta^3}{\theta^3+2}\cdot\dfrac{x^2+3x+(\theta^3+2\theta^2+\theta+2)}{(\theta+1)^{x+3}}\ ;x=0,1,2,...,\theta>0$

The expected values given by PLD, PAD and PID are also given in the table for ready comparison. It is very clear from the goodness of fit presented in tables 3, 4, and 5 that PRD provides a better fit over PLD, PAD and PID. Therefore PRD can be considered as an important over-dispersed discrete distribution to model over-dispersed count data from biological sciences.

Table 3. Summary of the datasets

Datasets	Mean	Variance	Skewness	Kurtosis
1	0.783333	1.236389	1.865895	3.672049
2	0.7500	1.294643	1.903268	3.670897
3	0.5475	1.122744	2.365317	5.694967

Table 4. Distribution of mistakes in copying groups of random digits, available in Kemp and Kemp (1965)

No. of errors per group	Observed Frequency	Expected Frequency				
		PD	PLD	PAD	PID	PRD
0	35	27.4	33.0	33.5	33.7	34.1
1	11	21.5	15.3	14.7	14.5	13.9
2	8	8.4	6.8	6.6	6.5	6.5
3	4	2.2	2.9	2.9	2.9	3.0
4	2	0.5	2.0	2.3	2.4	2.5
Total	60	60.0	60.0	60.0	60.0	60.0
ML estimate $\left(\hat{\theta}\right)$		0.7833	1.7434	2.0779	1.8643	2.4026
χ^2		7.98	2.20	1.40	1.33	1.02
d.f.		1	1	2	2	2
p-value		0.0047	0.1380	0.4966	0.5140	0.6005

Table 5. Distribution of Pyrausta nublilalis available in Beall(1940)

No. of insects	Observed Frequency	Expected Frequency				
		PD	PLD	PAD	PID	PRD
0	33	26.4	31.5	32.0	32.2	32.6
1	12	19.8	14.2	13.6	13.4	12.9
2	6	7.4	6.1	5.9	5.8	5.8
3	3	1.8	2.5	2.6	2.6	2.6
4	1	0.3	1.0	1.1	1.1	1.2
5	1	0.3	0.7	0.8	0.9	0.9
Total	56	56.0	56.0	56.0	56.0	56.0
ML estimate $\left(\hat{\theta}\right)$		0.7500	1.8081	2.1446	1.9186	2.4673
χ^2		4.87	0.53	0.24	0.20	0.09
d.f.		1	1	1	1	1
p-value		0.0273	0.4666	0.6242	0.6547	0.76290

Table 6. Distribution of number of Chromatid aberrations (0.2 g chinon 1, 24 hours) available in Janardan and Schaeffer (1977) & Loeschke and Kohler (1976)

No. of Chromatid aberrations	Observed Frequency	Expected Frequency				
		PD	PLD	PAD	PID	PRD
0	268	231.3	257.0	260.4	260.8	264.4
1	87	126.7	93.4	89.7	89.3	85.6
2	26	34.7	32.8	32.1	31.8	31.0
3	9	6.3	11.2	11.5	11.5	11.8
4	4	08	3.8	4.1	4.2	4.5
5	2	0.1	1.2	1.4	1.5	1.7
6	1	0.1	0.4	0.5	0.6	0.6
7+	3	0.1	0.2	0.3	0.3	0.4
Total	400	400.0	400.0	400.0	400.0	400
ML estimate $\left(\hat{\theta}\right)$		0.5475	2.3804	2.6594	2.3362	2.9260
χ^2		38.21	6.21	4.17	3.61	2.63
d.f.		2	3	3	3	3
p-value		0.0001	0.1018	0.2437	0.3067	0.4522

8. Concluding Remarks

In this paper a Poisson mixture of Rama distribution called Poisson-Rama distribution (PRD) has been suggested. The expressions of statistical constants including coefficients of variation, skewness, kurtosis and index of dispersion have been obtained and their behavior for varying values of parameter has been studied. It is observed that the obtained distribution is unimodal, has increasing hazard rate and over-dispersed. Both the method of moment and maximum likelihood estimation has been discussed for the estimation of parameter. A simulation study has been done to test the performance of maximum likelihood estimates. Finally, the goodness of fit of the proposed distribution and its comparison with other one parameter over-dispersed discrete distributions Poisson-Lindley distribution (PLD), Poisson-Akash distribution (PAD), and Poisson-Ishita distribution (PID) on three datasets from biological sciences has been presented. The result shows that the PRD provides greater flexibility in real over-dispersed count data.

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Conflicts of Interest

There is no conflict of interest.

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