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ARTICLE

Exploring Relationships and Inference Methods for Moments of Dual Generalized Order Statistics Derived from the Siddiqui-Dwivedi Distribution[1]

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Abstract

In this paper, recurrence relation for single and product moments of dual generalized order statistics (DGOS) from Siddiqui-Dwivedi **(**SD) distribution are obtained. Furthermore, we derive the minimum variance linear unbiased estimator (MVLUE) of the location and scale parameters of the kth lower record values (LVR) and order statistics (OS) for the distribution under consideration. In the section 4, a characterization result is presented.

Keywords: Moments, order statistics, Minimum variance linear unbiased estimator, SD distribution.

1. Introduction

Burkschat *et al*. (2003) introduced the concept of DGOS to provide a unified approach for dealing with descending ordered random variables (*DORV*), such as ROS and *LRV*. Ahsanullah (2004) was the first to characterize the uniform distribution using this framework.
Suppose $V'(1,b,c,k), V'(2,b,c,k), V'(3,b,c,k), \cdots V'(b,b,c,k)$, are a random sample from an absolutely continuous distribution function. Their joint pdf can be written as

$$k\left(\prod_{j=1}^{b-1}\gamma_j\right)\left(\prod_{i=1}^{b-1}G^c(v_i)g(v_i)\right)\left(G^{k-1}(v_b)g(v_b)\right) \tag{1.1}$$

where, $\gamma_j = k + (b - j)(c+1), \quad j = 1,2,..., b.$ within the cone $G^{-1}(1) > v_1 \geq v_2 \geq ... v_b > G^{-1}(0).$

If $c_i = 0, i = 1,2,...,b,\ k = 1,$ then $V'(a,b,c,k)$ reduce to the $(b-a+1)-th$ OS, $V_{b-a+1:b}$ from sample $V_1, V_2, ..., V_b$ and $c = -1,$ then $V'(a,b,c,k)$ reduce to $a-th,\ k-$LRV.

The pdf of $V'(a,b,c,k)$ is

$$g^{a,\Delta}(v) = \frac{D_{a-1}}{(a-1)!}[G(v)]^{(k+(b-a)(c+1))-1} g(v) I_c^{a-1}(F(v)) \tag{1.2}$$

and joint If pdf for $V'(a,\Delta)$ & $V'(s,\Delta),$ is $1 \le a < s \le b.$

$$f_{a,s,\Delta}(v,w) = \frac{D_{s-1}}{(a-1)!(s-a-1)!}[G(v)]^c g(v)\, I_c^{a-1}(F(v))$$
$$\times [h^c(G(w)) - h^c(G(v))]^{s-a-1}[G(w)]^{[k+(b-s)(c+1)]-1} g(w), v > w \tag{1.3}$$

where,

$$\Delta = b, c, k$$

$$D_{a-1} = \prod_{i=1}^{a} \gamma_i$$

$$h^c(v) = \begin{cases} -\dfrac{1}{c+1} v^{c+1} & , c \neq -1,\ v \in (0,] \\ -\log v & , c = -1 \end{cases}$$

and $I_c(v) = h^c(v) - h^c(1),\ v \in (0,1).$

Multiple researchers have employed the concept of DGOS in their studies. An excellent review on recurrence relations for moments of DGOS, is given by Pawlas & Szynal (2001), Mbah & Ahsanullah (2007), Athar & Faizan (2011), Athar *et al.* (2014), Nayabuddin (2013), Athar *et al.* (2015a,b), Nayabuddin & Salam (2017), Gupta & Anwar (2019), Khan, M.I. (2021), Athar *et al.* (2021), Kumar *et al.* (2021), Khatoon *et al.* (2021), among others.

In this study, we derived recurrence relations for the single and product moments of DGOS from the SD distribution. Additionally, we discussed various deductions and special cases, and presented and proved a theorem for characterizing the SD distribution.

According to Dwivedi *et al.* (2019), the pdf of the SD distribution is written as;

$$g(v) = (1-p)\,\theta^{p-1} v^{-p},\ 0 < p < 1, 0 < v < \theta \tag{1.4}$$

and corresponding df

$$G(v) = \theta^{p-1} v^{1-p}, \; 0 < p < 1, 0 < v < \theta \tag{1.5}$$

Given equations (1.4) and (1.5), we obtain;

$$G(v) = \frac{v}{1-p} g(v) \tag{1.6}$$

We will utilize equation (1.6) to obtain recurrence relations for the DGOS moments.

2. Single Moments

Theorem 2.1: Regarding distribution as defined in (1.4) and $c, b \in B, 1 \le a \le b, \; k > 0$

$$E[V'^j(a, \Delta)] = \frac{(1-p)\gamma_a}{(1-p)\gamma_a + j} E[V'^j(a-1, \Delta)] \tag{2.1}$$

and subsequently

$$E[V'^j(a, \Delta)] = \theta^j \prod_{l=1}^{a} \frac{(1-p)\gamma_l}{(1-p)\gamma_l + j} \tag{2.2}$$

Proof: From Athar *et al.*(2008), we have

$$E_v[\psi\{V'(a, \Delta)\}] - E_v[\psi\{V'(a-1, \Delta)\}]$$
$$= -\frac{D_{a-1}}{\gamma_a} \int_\alpha^\beta \psi'(v) [G(v)]^{\gamma_a} I_c^{a-1}(G(v))\, dv. \tag{2.3}$$

where, $\psi(v)$ is a borel measurable function of $v \in (\alpha, \beta)$

Let $\psi(v) = v^j$, then we have

$$E_v[\psi\{V'(a, \Delta)\}] - E_v[\psi\{V'(a-1, \Delta)\}]$$
$$= -j \frac{D_{a-1}}{\gamma_a} \int_\alpha^\beta v^{j-1} [F(v)]^{\gamma_a - 1} g_c^{a-1}(F(v))\, f(v) dv.$$

Considering equation (1.6), we obtain

$$E_v[V'^j(a,\Delta)] - E_v[V'^j(a-1,\Delta)]$$

$$= -\frac{j}{1-p}\frac{D_{a-1}}{\gamma_a}\int_\alpha^\beta v^{j-1}[G(v)]^{\gamma_a-1} I_c^{a-1}(G(v))\, g(v)dv.$$

and hence the (2.1).

Since $V_{0,n,m,k} = \theta$, the maximum of V in new distribution, then we have

$$E_v[V'^j(1,\Delta)] = \frac{(1-p)\gamma_1}{(1-p)\gamma_1 + j}\theta^j \tag{2.4}$$

Equation (2.2) can be derived by applying (2.1) repeatedly and using equation (2.4) as the initial value.

Remark 2.1: The recurrence relation for the single moments of order statistics (at $c = 0, k = 1$) is given as;

$$E_v(V'^j_{b-a+1:b}) = \frac{(b-a+1)}{(1-p)(b-a+1)+j}E_v(V'^j_{b-a+2:b})$$

$$E_v(V'^j_{b-a+1:b}) = \frac{\Gamma(b-a+1)(1-p)+j}{\Gamma(b+1)(1-p)+j}\frac{\Gamma(b+1)}{\Gamma(b-a+1)}\theta^j$$

Replace $b-a+1$ by a, we get

$$E_v(V'^j_{a:b}) = \frac{a}{(1-p)a+j}E_v(V'^j_{a+1:b})$$

$$E_v(V'^j_{a:b}) = \frac{\Gamma a(1-p)+j}{\Gamma(b+1)(1-p)+j}\frac{\Gamma(b+1)}{\Gamma a}\theta^j \tag{2.5}$$

Expression (2.5) may also be re written as

$$E_v(V'^j_{a:b}) = D_{a:b}\; B\left(\frac{j}{1-p}+a, b-a+1\right)\theta^j$$

as obtained by Dwivedi *et al.* (2019).

Remark 2.2: The recurrence relation for the single moments of the $k-th$ lower record (at $c = -1$)

will be

$$E_v(V'^{(k)}_a)^j = \frac{(1-p)k}{(1-p)k+j}\theta^j E_v(V'^{(k)}_{a-1})^j$$

$$E_v(V'^{(k)}_a)^j = \theta^j\left(\frac{(1-p)k}{(1-p)k+j}\right)^a$$

Table 1. Mean of order statistics

a	$\theta=2\,\&\,p=0.5$	$\theta=3\,\&\,p=0.65$	$\theta=2\,\&\,p=0.75$	$\theta=3\,\&\,p=0.95$
1	0.666666	0.777777	0.400000	0.142857
1	0.333333	0.320261	0.133333	0.012987
2	1.000000	1.235294	0.666666	0.272727
1	0.200000	0.164036	0.057142	0.001693
2	0.600000	0.632711	0.285714	0.035573
3	1.200000	1.536585	0.857142	0.001693
1	0.133333	0.095687	0.028571	0.000282
2	0.400000	0.369081	0.142857	0.005928
3	0.800000	0.896341	0.428571	0.065217
4	1.333333	1.750000	1.000000	0.500000
1	0.095238	0.060892	0.015873	5.646e-05
2	0.285714	0.234870	0.079365	0.001185
3	0.571428	0.570399	0.238095	0.013043
4	0.952381	1.113636	0.555555	0.100000
5	1.428571	1.909091	1.111111	0.600000
1	0.071428	0.041249	0.009523	1.303e-05
2	0.214285	0.159105	0.047619	0.000273
3	0.428571	0.386399	0.142857	0.003010
4	0.714285	0.754398	0.333333	0.023076
5	1.071429	1.293255	0.666666	0.138461
6	1.500000	2.032258	1.200000	0.692307
1	0.055555	0.029293	0.006060	3.378e -06
2	0.166666	0.112988	0.030303	7.095e-05
3	0.333333	0.274399	0.090909	0.000780
4	0.555555	0.535732	0.212121	0.005982
5	0.833333	0.918398	0.424242	0.035897
6	1.166667	1.443198	0.763636	0.179487
7	1.555556	2.130435	1.272727	0.777777

As observed in Table 2.1, the well-known property of order statistics, $\sum_{i=1}^{b} E_v(V_{i:b}) = bE_v(V)$ as described by David and Nagaraja (2003), is satisfied.

Table 2. The mean value of k^{th} lower record

a	$\theta=2, k=2$ $p=0.50$	$\theta=3, k=3$ $p=0.65$	$\theta=4, k=4$ $p=0.75$	$\theta=5, k=5$ $p=0.95$
1	1.000000	1.536585	2.000000	2.777778
2	0.500000	0.787031	1.000000	1.543210
3	0.250000	0.403113	0.500000	0.857338
4	0.125000	0.206472	0.250000	0.476299
5	0.062500	0.105754	0.125000	0.264610
6	0.031250	0.054166	0.062500	0.147006
7	0.015625	0.027744	0.031250	0.081669
8	0.007812	0.014210	0.015625	0.045372
9	0.003906	0.007278	0.007812	0.025206
10	0.001953	0.003727	0.003906	0.014003

3. Product Moments

Theorem 3.1: For the distribution as given in (1.4). Fix k (a positive integer)and for $b \in B, c \in R$, $1 \le a < s \le b$

$$E_v[V'^j(a,\Delta).V'^j(s,\Delta)] = \frac{(1-p)\gamma_s}{(1-p)\gamma_s + j} E_v[V'^j(a,\Delta).V'^j(s-1,\Delta)] \quad (3.1)$$

$$= \theta^{i+j} \prod_{q=a+1}^{s} \left(\frac{(1-p)\gamma_q}{(\gamma_q + j)(1-p)} \right) \prod_{l=1}^{a} \left(\frac{(1-p)\gamma_l}{(\gamma_l + i + j)(1-p)} \right) \quad (3.2)$$

Proof: According to Athar (2008), we have

$$E_v[\psi\{V'(a,\Delta).V'(s,\Delta)\}] - E_v[\xi\{V'(a,\Delta).V'(s-1,\Delta)\}]$$

$$= -\frac{D_{s-1}}{\gamma_s(a-1)!(s-a-1)!} \int\int_{\alpha \le w < v \le \beta} \frac{\partial}{\partial w} \psi(v,w)[G(v)]^c g(v)\, I_c^{a-1}(G(v))$$

$$\times [h_c(G(w)) - h_c(G(v))]^{s-a-1} [G(w)]^{\gamma_s - 1} g(w) dvdw. \quad (3.3)$$

Let $\psi(v,w) = \psi_2(w).\psi_1(v) = w^j.v^i$ in (3.3), then in (1.6), we have

$$E_v[V'^i(a,\Delta).V'^j(s,\Delta)] - E_v[V'^i(a,\Delta).V'^j(s-1,\Delta)]$$

$$= -\frac{jD_{s-1}}{(1-p)\gamma_s(a-1)!(s-a-1)!} \int_0^\theta \int_0^v v^i w^j [G(v)]^c g(v)\, I_c^{a-1}(G(v))$$

$$\times [h_c(G(w)) - h_c(G(v))]^{s-a-1} [G(w)]^{\gamma_s - 1} g(w) dvdw.$$

which leads to (3.1).

(3.2) follows by, writing (3.1) repeatedly.

Remark 3.1: The recurrence relation for the product moments of order statistics (at $c = 0, k = 1$) is

$$E_v(V'^{i,j}_{b-a+1,b-s+1:b}) = \frac{(b-s+1)(1-p)}{(b-s+1)(1-p)+j} E_v(V'^{i,j}_{b-a+1,b-s+2:b})$$

$$= \theta^{i+j} \frac{\Gamma(b+1)}{\Gamma(b+1-s)} \frac{\Gamma(b+1-s)(1-p)+j}{\Gamma(b+1-a)(1-p)+j} \frac{\Gamma(b+1-a)(1-p)+i+j}{\Gamma(b+1+i+j)}$$

Using $b+1-a$ as s and $b+1-s$ as a, we have

$$E_v(V'^{i,j}_{a,s:b}) = \frac{(1-p)a}{a(1-p)+j} E_v(V'^{i,j}_{a,s-1:b})$$

$$= \theta^{i+j} \frac{\Gamma(n+1)}{\Gamma(r)} \frac{\Gamma(1-p)r+j}{\Gamma(1-p)s+j} \frac{\Gamma(1-p)s+i+j}{\Gamma(b+1+i+j)}$$

Remark 3.2: Recurrence relation for product moments of $k-th$ record values (at $c=-1$) is

$$E_v[(V'^{(k)}_a)^i.(V'^{(k)}_s)^j] = \frac{(1-p)k}{(1-p)k+j} E_v[(V'^{(k)}_a)^i.(V'^{(k)}_{s-1})^j]$$

$$= \theta^{i+j} \left(\frac{(1-p)k}{(1-p)k+j}\right)^{s-a} \left(\frac{(1-p)k}{(1-p)k+i+j}\right)^{a}$$

Remark 3.1: At $i=0$, the result has reduced to single moment. Refer to Theorem (2.1).

Table 3. Variance and covariance of order statistics

b	s	a	$p=0.50, \theta=2$	$p=0.65, \theta=3$	$p=0.75, \theta=2$	$p=0.95, \theta=3$
1	1	1	0.355556	0.735487	0.284444	0.199104
2	1	1	0.155555	0.244950	0.071111	0.0102843
	2	1	0.111111	0.209321	0.071111	0.0168662
		2	0.333333	0.807381	0.355555	0.3541913
3	1	1	0.074285	0.092729	0.020977	0.000726
	2	1	0.070476	0.104469	0.027309	0.001363
		2	0.211428	0.402954	0.136549	0.028634
	3	1	0.045714	0.078705	0.023747	0.002055
		2	0.137142	0.303580	0.118738	0.043162
		3	0.274285	0.737266	0.356215	0.474788
4	1	1	0.039365	0.040106	0.007264	6.621e-05
	2	1	0.041904	0.050436	0.010463	0.000128
		2	0.125714	0.19454	0.052319	0.002683
	3	1	0.036190	0.050427	0.011997	0.000228
		2	0.108571	0.194504	0.059987	0.004802
		3	0.217142	0.472367	0.179962	0.052829
	4	1	0.022222	0.035179	0.009523	0.000321
		2	0.066666	0.135691	0.04761	0.006737
		3	0.133333	0.329537	0.142857	0.074110
		4	0.222222	0.643382	0.333333	0.56818
5	1	1	0.022675	0.019281	0.002856	7.363e-06
	2	1	0.025699	0.025716	0.004334	1.431e-05
		2	0.077097	0.099191	0.021673	0.000301
	3	1	0.024943	0.028825	0.005544	2.672e-05
		2	0.074829	0.111183	0.027723	0.000561
		3	0.149659	0.270016	0.083170	0.006172
	4	1	0.020408	0.026750	0.005833	4.568e-05
		2	0.061224	0.103179	0.029168	0.000959
		3	0.122449	0.250579	0.087505	0.010553
		4	0.204081	0.489225	0.204178	0.080909
	5	1	0.012093	0.017714	0.004341	6.022e-05
		2	0.036281	0.068325	0.021706	0.001264
		3	0.072562	0.16593	0.065120	0.013913
		4	0.120937	0.323966	0.151946	0.106666
		5	0.181405	0.555371	0.303893	0.640000

Table 4. Variance and covariance value of k-th lower record

s	a	$\theta = 2, k = 2, p = 0.5$	$\theta = 3, k = 3, p = 0.65$	$\theta = 4, k = 4, p = 0.75$	$\theta = 5, k = 5, p = 0.95$
1	1	0.333333	0.737266	1.333333	1.777778
2	1	0.166666	0.377624	0.666666	0.355556
	2	0.194444	0.447230	0.777777	0.268642
3	1	0.083333	0.193417	0.333333	0.071111
	2	0.097222	0.229069	0.388888	0.053728
	3	0.085648	0.204706	0.342592	0.032693
4	1	0.041666	0.099067	0.166666	0.014222
	2	0.048611	0.117328	0.194444	0.010745
	3	0.042824	0.104849	0.171296	0.006538
	4	0.033757	0.083784	0.135030	0.003746
5	1	0.020833	0.050741	0.083333	0.002844
	2	0.024305	0.060094	0.097222	0.002149
	3	0.02141	0.053703	0.085648	0.001307
	4	0.016878	0.042913	0.067515	0.000749
	5	0.012554	0.032336	0.050218	0.000421

4. Characterization

Theorem 4.1: If V is a continuous random variable with cdf and pdf. for $0 < F(v) < 1$ with all $v > 0$, then

$$E_v[V(s,\Delta) \mid V(a,\Delta) = v] = q^*_{s|l}\, v, \quad l = a, a+1$$

where $$q^*_{s|l} = \prod_{j=a+1}^{s} \frac{(1-p)\gamma_j}{(1-p)\gamma_j + 1}$$

If

$$G(v) = \left(\frac{v}{\theta}\right)^{1-p}, \quad 0 < p < 1, \quad 0 < v < \theta \tag{4.1}$$

Proof: For, $s \geq a+1$

$$g_{s|a}(v) = E_v[V(s,\Delta) \mid V(a,\Delta) = v]$$

$$= \frac{D_{s-1}}{D_{a-1}(s-a-1)!(c+1)^{s-a-1}} \int_0^v w \left(\frac{G(w)}{G(v)}\right)^{\gamma_s - 1} \left[1 - \left(\frac{G(w)}{G(v)}\right)^{(c+1)}\right]^{s-a-1} \frac{g(w)}{G(v)}\, dv \tag{4.2}$$

Let $t = \left(\frac{G(w)}{G(v)}\right) = \left(\frac{w}{v}\right)^{1-p}$, $w = 0, t = 0 \quad w = v, t = 1, \; w = v\, t^{\frac{1}{1-p}}$ (4.3)

Applying (4.2)& (4.3) gives;

$$= \frac{D_{s-1}}{D_{a-1}(s-a-1)!(c+1)^{s-a-1}} \int_0^1 t^{\gamma_s - 1}[1-t^{(c+1)}]^{s-a-1} t^{\gamma_s + \frac{1}{1-p} - 1}\, dv \qquad (4.4)$$

Set $t^{c+1} = z$ in (4.4), we get

$$= \frac{D_{s-1}}{D_{a-1}(s-a-1)!(c+1)^{s-a-1}} \int_0^1 Z^{\frac{\gamma_s + \frac{1}{1-p} - 1}{c+1}} (1-z)^{s-a-1} dz \qquad (4.5)$$

After simplification, we get

$$g_{s|a}(v) = v \prod_{j=a+1}^{s} \frac{(1-p)\gamma_j}{(1-p)\gamma_j + 1} \qquad (4.6)$$

$$g_{s|a+1}(v) = v \prod_{j=a+2}^{s} \frac{(1-p)\gamma_j}{(1-p)\gamma_j + 1}$$

$$g_{s|a+1}(v) - g_{s|a}(v) = \frac{v}{1-p} \frac{1}{\gamma_{a+1}} \prod_{j=a+1}^{s} \frac{(1-p)\gamma_j}{(1-p)\gamma_j + 1} \qquad (4.7)$$

$$g'_{s|a}(v) = \prod_{j=a+1}^{s} \frac{(1-p)\gamma_j}{(1-p)\gamma_j + 1} \qquad (4.8)$$

Using (4.7) and (4.8) gives;

$$\frac{1}{\gamma_{a+1}} \frac{g'_s|_r (v)}{g_s|_{a+1}(v) - g_s|_a (v)} = \frac{1-p}{v},\ 0 < p < 1,\ 0 < v < \theta \qquad (4.9)$$

So that;

$$G(v) = \left(\frac{v}{\theta}\right)^{1-p},\ 0 < p < 1, \quad 0 < v < \theta \qquad (4.10)$$

and hence the result.

5. Application

Finding the minimum variance linear unbiased estimates (MVLUE) of the location and scale parameters is an intriguing use case for this work. Although the Llyod's (1952) technique can be applied, numerical methods were utilized to obtain these estimates because it was challenging to obtain the inverse of the variance-covariance matrix in closed form.

Assume the random variables have the location parameter μ and scale parameter σ. Utilizing Lloyd's method, the MVLUE of θ as $\hat{\theta} = (M'V_*^{-1}w)(M'V_*^{-1}M)^{-1}$ where $V_* = (V_{*_{i,j}})$ is the variance of the $i-th$ and $j-th$ concomitants, V_*^{-1} is the inverse of the matrix V_*, w' is the observed value of the vector $W' = W_{d[1,\Delta]}, W_{d[2,\Delta]}, ..., W_{d[b,\Delta]}$, $\mu = 0$ and $\sigma = 1$ M and θ are defined as:

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \mu_{d[1,n,m,k]} & \mu_{d[2,n,m,k]} & \mu_{d[3,n,m,k]} & \mu_{d[4,n,m,k]} \end{bmatrix}, \ \theta' = [\mu \quad \sigma].$$

Table 5. Coefficients of MVLUE of μ & σ for SD distribution based on order statistics with $b = 4$

(p,θ)	Estimate	Coefficients			
(0.15,2)	$\hat{\mu}$	2.996976	-2.167600	-0.670221	0.840844
	$\hat{\sigma}$	-1.950831	1.411721	0.440452	0.098658
(0.25,3)	$\hat{\mu}$	0.916221	0.3177101	0.198384	-0.432315
	$\hat{\sigma}$	-0.407204	-0.141038	-0.088502	0.636744
(0.50,2)	$\hat{\mu}$	0.902482	0.210456	0.096786	-0.209725
	$\hat{\sigma}$	-0.680951	-0.156311	-0.070733	0.907996
(0.65,3)	$\hat{\mu}$	0.907621	0.148589	0.050747	-0.106957
	$\hat{\sigma}$	-0.518642	-0.084907	-0.028998	0.6325471

Table 6. Coefficients of MVLUE of μ and σ for SD distribution based on $k-th$ lower record values with $b = 4$

(p,θ,k)	Estimate	Coefficients			
(0.15,2,2)	$\hat{\mu}$	0.185358	-0.182810	-0.277135	1.645304
	$\hat{\sigma}$	0.6975193	0.488006	0.578861	-1.764386
(0.25,3,3)	$\hat{\mu}$	-0.426509	0.612058	-1.815154	2.629605
	$\hat{\sigma}$	0.568799	-0.262818	1.331230	-1.637212
(0.50,2,2)	$\hat{\mu}$	0.088124	-0.054916	-0.218381	1.361421
	$\hat{\sigma}$	0.777434	0.548821	0.911495	-2.237751
(0.65,3,3)	$\hat{\mu}$	-0.199829	-0.268203	1.098468	0.369564
	$\hat{\sigma}$	0.619973	0.570215	-0.803754	-0.386434

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Conflicts of Interest

The authors declare no conflict of interest.

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