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ARTICLE

Improved estimation of population mean based on hybrid exponentially weighted moving average

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Abstract

In sampling theory, the researchers are often dependent on estimators that use only current sample data to estimate population parameters. However, the hybrid exponentially weighted moving average (HEWMA) approach incorporates both current and past sample information and helps increasing the efficiency of the estimators. This enables us to develop an improved estimation procedure for temporal surveys based on HEWMA. We develop memory-type log estimator of population mean based on HEWMA under simple random sampling (SRS). We derive the bias and mean square error (MSE) of the developed estimator to the first-order approximation. The efficiency conditions are established by comparing the MSE of the proposed estimator with the MSE of the available traditional and memory-type estimators. To validate our theoretical findings, we conduct a simulation study utilizing hypothetically drawn population. A real data illustration of the developed methods is also presented. The findings demonstrate that our approach integrates past and present sample information and enhances the estimators' efficacy.
Keywords: Hybrid exponentially weighted moving average; Mean square error; Bias; Efficiency.

1. Introduction

Sampling theory plays a crucial role in several fields, including statistics, economics, sociology, and epidemiology, among others. It gives a framework for making inferences about a population based on the sample chosen from that population. The conventional sampling methods often depend solely on the information obtained from the sample itself to drawn inferences about the population parameters. However, in many real-life situations, the auxiliary information may be considered to improve the efficiency and accuracy of the estimation procedures.

In sampling theory, the utilization of auxiliary information also provides many advantages, namely, reducing sampling costs, decreasing the sample size required to achieve a desired level of precision, saving time and resources, among others. By auxiliary information with relevant auxiliary variables, estimators can better capture the underlying population parameters, leading to more reliable inference. Furthermore, incorporating auxiliary information allows for the construction of more robust sampling designs adapted to specific study objectives and population characteristics. Researchers may strategically choose auxiliary variables that are correlated with the study variables, thereby improving the efficiency of the estimation procedures. When the study variable is positively correlated with the auxiliary variable, Cochran (1940) suggested to employ the ratio estimator prescribed hereunder as

$$t_r = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right) \tag{1.1}$$

where $\bar{x} = n^{-1}\sum_{i=1}^{n} x_i$ and $\bar{y} = n^{-1}\sum_{i=1}^{n} y_i$ are the sample means of the auxiliary variable x and study variable y, respectively. Also, $\bar{X} = N^{-1}\sum_{i=1}^{N} x_i$ is the population mean of auxiliary variable x. The MSE of the ratio estimator t_r is given by

$$MSE(t_r) = q\bar{Y}^2(C_y^2 + C_x^2 - 2\rho_{xy}C_xC_y) \tag{1.2}$$

where $q = 1/n$, $\bar{Y} = N^{-1}\sum_{i=1}^{N} y_i$ is the population mean of study variable y, (C_x, C_y) are the population coefficient of variations of variables (x, y), respectively.

Bhushan and Gupta (2015) suggested the log type estimator for population mean as

$$t_{bg} = \bar{y}\left\{1 + \log\left(\frac{\bar{x}}{\bar{X}}\right)\right\}^{\beta} \tag{1.3}$$

where β is a properly selected scalar.
The minimum MSE at optimum value of $\beta_{(opt)} = -\rho_{xy}(C_y/C_x)$ is given by

$$min.MSE(t_{bg}) = q\bar{Y}^2C_y^2(1-\rho_{xy}^2) \tag{1.4}$$

Bhushan and Gupta (2015) also suggested an improved version of log estimator for population mean given as

$$t_s = \alpha\bar{y}\left\{1 + \log\left(\frac{\bar{x}}{\bar{X}}\right)\right\}^{\beta} \tag{1.5}$$

where α and β are properly selected scalars.
The minimum MSE of the estimator t_s at optimum value of $\alpha_{(opt)} = B/A$ is given below as

$$min.MSE(t_s) = \bar{Y}^2\left(1 - \frac{B^2}{A}\right) \tag{1.6}$$

where $A = 1 + qC_y^2 + 2\beta(\beta-1)qC_x^2 + 4\beta q\rho_{xy}C_xC_y$ and $B = 1 + \left(\frac{\beta^2}{2} - \beta\right)qC_x^2 + \beta q\rho_{xy}C_xCy$.

In several sectors such as finance, signal processing, and data science, the accurate estimation of parameters from time-series data is of great importance. Conventional estimation procedures often face challenges in efficiently handling large amounts of data, while, maintaining high accuracy and adaptability to changing conditions. Memory type estimators incorporate information from both present and past observations and give a promising solution to these challenges by capturing the underlying trends and dynamics of the data stream.

Among memory-type estimators, the exponentially weighted moving average (EWMA) has earned a widespread popularity for its simplicity and effectiveness in capturing recent data trends, while, reducing the impact of past observations. Noor-Ul-Amin (2019) introduced memory-type ratio and product estimators of population mean employing EWMA for temporal surveys under SRS. Aslam et al. (2020) and Aslam et al. (2021) developed memory-type ratio and product estimators of population mean based on EWMA statistic under stratified sampling and ranked-based sampling, respectively. Qureshi et al. (2022) proposed memory-type ratio and product estimators for population variance employing EWMA statistics for temporal surveys. Bhushan et al. (2023) suggested memory-type log estimators for temporal surveys based on EWMA statistic.

The conventional EWMA-based estimators may suffer from limitations in adaptability to dynamic environments and may not fully leverage the available information in the data. To address these limitations, a new approach known as hybrid exponentially weighted moving average (HEWMA) has emerged which combines the advantages of EWMA with other memory-based methods to achieve superior performance in terms of accuracy, efficiency, and adaptability. HEWMA provides a flexible framework for incorporating both short-term and long-term memory into the estimation procedure allowing for better capture of complex data patterns and dynamics. Noor-Ul-Amin (2020) introduced the memory-based ratio and product estimators for population mean utilizing HEWMA for temporal surveys under SRS. Bhushan et al. (2022) evaluated the performance of the memory-type log estimators employing HEWMA. In this paper, we develop HEWMA-based efficient memory-type log estimator under SRS.

In the next section, we review the existing HEWMA-based memory-type estimators and their properties. In Section 3, we propose the efficient memory-type log estimators and explore the theoretical foundations of the proposed estimators, properties, and advantages over existing conventional and HEWMA-based estimators. In Section 4, the mathematical conditions are reported under which the proposed estimators will dominate the existing estimators. We investigate the practical implementation of proposed memory-type estimators, including algorithmic details and interpretation of findings in Section 5. A real data illustration is also presented in Section 6. The manuscript is concluded in Section 7.

2. Review of memory-type estimators

The HEWMA statistic presents a novel approach in statistical analysis, particularly in the realm of process monitoring and control charting. Developed by Haq (2013), HEWMA builds upon the foundation of the conventional EWMA statistic pioneered by Robert (1959). HEWMA combines the strengths of EWMA with additional memory-based methods, allowing for the incorporation of both current and past information in the estimation process. This hybrid approach improves the adaptability and efficiency of the statistic enabling more robust detection of changes in the underlying process mean over time. Let $X_1, X_2, ..., X_n$ denote the independent and identically distributed random variables. Using these random variables, we define the sequence $HE_1, HE_2, ..., HE_n$ by employing the following recursive expressions.

$$E_t = \lambda_2 \bar{X}_t + (1-\lambda_2)E_{t-1} \quad 0 < \lambda_2 \leq 1, \qquad HE_t = (1-\lambda_1)HE_{t-1} + \lambda_1 E_t \quad 0 < \lambda_1 \leq 1 \tag{2.1}$$

where the scalars λ_i, $i = 1, 2$, are properly selected coefficients. In addition, the EWMA and HEWMA statistics are symbolized by E_t and HE_t, respectively. The primary amounts of these statistics are regarded as the expected mean which may be computed from preliminary data such as a pilot survey. For this study, these initial values are set to zero, i.e., $HE_0 = E_0 = 0$. Haq (2013) calculated the mean and variance of the HEWMA statistic. However, Haq (2016) subsequently identified errors in the expressions derived in the previous work. As a result, Haq (2016) provided corrected expressions for the mean and variance of the HEWMA statistic which are given by:

$$E(HE_t) = \bar{Y}, \qquad V(HE_t) = \frac{\lambda_1^2\lambda_2^2}{(\lambda_1-\lambda_2)^2}\left\{\begin{array}{c} \frac{(1-\lambda_1)^2(1-(1-\lambda_1)^{2t})}{1-(1-\lambda_1)^2} + \frac{(1-\lambda_2)^2(1-(1-\lambda_2)^{2t})}{1-(1-\lambda_2)^2} \\ -\frac{2(1-\lambda_1)(1-\lambda_2)(1-(1-\lambda_1)^t(1-\lambda_2)^t)}{1-(1-\lambda_1)(1-\lambda_2)} \end{array}\right\}\frac{\sigma_y^2}{n} \tag{2.2}$$

For $t \geq 1$, σ_y^2 denotes the variance of the variable y. The limiting expression for the variance is given as

$$V(HE_t) = \frac{\lambda_1^2\lambda_2^2}{(\lambda_1-\lambda_2)^2}\left\{\frac{(1-\lambda_1)^2}{1-(1-\lambda_1)^2}+\frac{(1-\lambda_2)^2}{1-(1-\lambda_2)^2}-\frac{2(1-\lambda_1)(1-\lambda_2)}{1-(1-\lambda_1)(1-\lambda_2)}\right\}\frac{\sigma^2}{n} \tag{2.3}$$

$$V(HE_t) = \frac{\lambda_1^2\lambda_2^2}{(\lambda_1-\lambda_2)^2}\delta\frac{\sigma_y^2}{n} \tag{2.4}$$

where $\delta = \frac{(1-\lambda_1)^2}{1-(1-\lambda_1)^2}+\frac{(1-\lambda_2)^2}{1-(1-\lambda_2)^2}-\frac{2(1-\lambda_1)(1-\lambda_2)}{1-(1-\lambda_1)(1-\lambda_2)}$. It is remarkable that the value of δ will be substituted with δ_1 as specified by

$$\delta_1 = \left\{\frac{(1-\lambda_1)^2(1-(1-\lambda_1)^{2t})}{1-(1-\lambda_1)^2}+\frac{(1-\lambda_2)^2(1-(1-\lambda_2)^{2t})}{1-(1-\lambda_2)^2}-\frac{2(1-\lambda_1)(1-\lambda_2)(1-(1-\lambda_1)^t(1-\lambda_2)^t)}{1-(1-\lambda_1)(1-\lambda_2)}\right\} \tag{2.5}$$

Noor-ul-Amin (2020) utilized the HEWMA statistic to introduce memory-type estimation procedures for population mean under SRS. The variable y has HEWMA statistic as

$$E_{ty} = \lambda_2\bar{y}_t+(1-\lambda_2)E_{ty-1}, \qquad Z_t = \lambda_1 E_{ty}+(1-\lambda_1)Z_{t-1}, \tag{2.6}$$

and the variable x has HEWMA statistic as

$$E_{tx} = \lambda_2\bar{x}_t+(1-\lambda_2)E_{tx-1}, \qquad Q_t = \lambda_1 E_{tx}+(1-\lambda_1)Q_{t-1}. \tag{2.7}$$

The statistics Q_t and Z_t are unbiased estimators for the population means $\bar{X}$ and $\bar{Y}$, respectively. For more details, see Appendix A.

Employing the HEWMA statistics Q_t and Z_t, Noor-ul-Amin (2020) introduced the memory-type ratio estimator under SRS as

$$t_r^m = Z_t\left(\frac{\bar{X}}{Q_t}\right). \tag{2.8}$$

To establish the properties of the memory-type estimators, we assume that $Z_t = \bar{Y}(1+e_0)$ and $Q_t = \bar{X}(1+e_1)$ such that $E(e_0) = E(e_1) = 0$ and $E(e_0^2) = q\zeta C_y^2$, $E(e_1^2) = q\zeta C_x^2$, $E(e_0e_1) = q\zeta\rho_{xy}C_xC_y$, where $\zeta = \delta\{(\lambda_1\lambda_2)^2/(\lambda_1-\lambda_2)^2\}$.
The MSE of the ratio estimator approximated to the first order is expressed as

$$MSE(t_r^m) = q\bar{Y}^2\zeta(C_y^2+C_x^2-2\rho_{xy}C_xC_y). \tag{2.9}$$

Bhushan et al. (2022) presented the memory-type log estimator within the framework of SRS as

$$t_{bg}^m = Z_t\left[1+\log\left(\frac{Q_t}{\bar{\bar{X}}}\right)\right]^{\beta_1} \tag{2.10}$$

where β_1 is a properly selected scalar.
The optimum MSE at $\beta_{1(opt)} = -\rho_{xy}C_y/C_x$ of the estimator t_{bg}^m is provided below

$$MSE(t_{bg}^m)_{(opt)} = q\zeta\bar{Y}^2C_y^2(1-\rho_{xy}^2) \tag{2.11}$$

3. Proposed memory-type estimators

The memory-type estimators based on HEWMA statistics represent a powerful tool for accurate parameter estimation in dynamic systems. Their ability to handle non-linear relationships, skewed distributions, and efficiently manage memory makes them well-suited for a wide range of applications, from financial forecasting to industrial process control. In this paper, we propose HEWMA-based improved memory-type log estimator for population mean under SRS as

$$t_s^m = \alpha_1 Z_t \left\{1 + \log\left(\frac{Q_t}{\bar{X}}\right)\right\}^{\beta_1} \tag{3.1}$$

where α_1 and β_1 are the properly selected scalars.

Remark 3.1. *For $\alpha_1 = 1$, the proposed estimator t_s^m reduces into memory-type log estimator t_{bg}^m envisaged by Bhushan et al. (2022).*

Theorem 3.1. *The bias and minimum MSE of the proposed memory-type estimator are given to the first order approximation as*

$$Bias(t_s^m) = \bar{Y}\left[\alpha_1\left\{1 + \left(\frac{\beta_1^2}{2} - \beta_1\right)\zeta q C_x^2 + \beta_1 \zeta q \rho_{xy} C_x C_y\right\} - 1\right] \tag{3.2}$$

$$min.MSE(t_s^m) = \bar{Y}^2\left(1 - \frac{Q_1^2}{P_1}\right) \tag{3.3}$$

where $P_1 = 1 + \zeta q C_y^2 + 2\beta_1(\beta_1 - 1)\zeta q C_x^2 + 4\beta_1 \zeta q \rho_{xy} C_x C_y$ *and* $Q_1 = 1 + \left(\frac{\beta_1^2}{2} - \beta_1\right)\zeta q C_x^2 + \beta_1 \zeta q \rho_{xy} C_x C_y$.

Proof. Utilizing the notations defined in previous section, the suggested memory-based log estimator for the population mean in SRS may be expressed as

$$\begin{aligned} t_s^m &= \alpha_1 Z_t \left\{1 + \log\left(\frac{Q_t}{\bar{X}}\right)\right\}^{\beta_1} \\ &= \alpha_1 \bar{Y}(1 + e_0)\left\{1 + \log\left(\frac{\bar{X}(1 + e_1)}{\bar{X}}\right)\right\}^{\beta_1} \\ &= \alpha_1 \bar{Y}(1 + e_0)\left\{1 + \left(e_1 - \frac{e_1^2}{2}\right)\right\}^{\beta_1} \end{aligned} \tag{3.4}$$

Employ Taylor series expansion, multiply, and ignore the error terms having power greater than two, we get

$$\begin{aligned} t_s^m &= \alpha_1 \bar{Y}(1 + e_0)\left\{1 + \beta_1\left(e_1 - \frac{e_1^2}{2}\right) + \frac{\beta_1(\beta_1 - 1)}{2} e_1^2\right\} \\ &= \alpha_1 \bar{Y}\left\{1 + e_0 + \beta_1 e_1 + \beta_1 e_0 e_1 + \left(\frac{\beta_1^2}{2} - \beta_1\right) e_1^2\right\} \end{aligned}$$

Subtract $\bar{Y}$ on both side in the above expression, we get

$$t_s^m - \bar{Y} = \bar{Y}\left[\alpha_1\left\{1 + e_0 + \beta_1 e_1 + \beta_1 e_0 e_1 + \left(\frac{\beta_1^2}{2} - \beta_1\right) e_1^2\right\} - 1\right] \tag{3.5}$$

Taking expectation on both side to (3.5), we get

$$Bias(t_s^m) = \bar{Y}\left[\alpha_1\left\{1+\left(\frac{\beta_1^2}{2}-\beta_1\right)\zeta q C_x^2+\beta_1\zeta q\rho_{xy}C_xC_y\right\}-1\right] \tag{3.6}$$

Squaring and taking expectation on both side to (3.5), we get the MSE of proposed memory-type estimator t_s^m as:

$$MSE(t_s^m) = \bar{Y}^2\begin{bmatrix} 1+\alpha_1^2\left\{\ 1+\zeta q C_y^2+2\beta_1(\beta_1-1)\zeta q C_x^2+4\beta_1\zeta q\rho_{xy}C_xC_y\ \right\} \\ -2\alpha_1\left\{1+\left(\frac{\beta_1^2}{2}-\beta_1\right)\zeta q C_x^2+\beta_1\zeta q\rho_{xy}C_xC_y\right\}\end{bmatrix}$$

$$MSE(t_s^m) = \bar{Y}^2(1+\alpha_1^2P_1-2\alpha_1Q_1) \tag{3.7}$$

where $P_1 = 1+\zeta q C_y^2+2\beta_1(\beta_1-1)\zeta q C_x^2+4\beta_1\zeta q\rho_{xy}C_xC_y$ and $Q_1 = 1+\left(\frac{\beta_1^2}{2}-\beta_1\right)\zeta q C_x^2+\beta_1\zeta q\rho_{xy}C_xC_y$. Minimizing (3.7) with respect to α_1, we get

$$\alpha_{1(opt)} = \frac{Q_1}{P_1}$$

Putting the optimum value of $\alpha_{1(opt)}$ in (3.7), we get the minimum MSE as

$$min.MSE(t_s^m) = \bar{Y}^2\left(1-\frac{Q_1^2}{P_1}\right)$$

□

4. Mathematical conditions

In this section, we obtain the mathematical conditions by comparing the MSE expressions of the proposed and available traditional and memory-type estimators.

- Comparing the proposed estimator t_s^m with the conventional ratio estimator t_r, we get

$$min.MSE(t_s^m) < MSE(t_r) \implies \frac{Q_1^2}{P_1} > 1-q\left(C_y^2+C_x^2-2\rho_{xy}C_xC_y\right)$$

- Comparing the proposed estimator t_s^m with the conventional log estimator t_{bg}, we get

$$min.MSE(t_s^m) < MSE(t_{bg}) \implies \frac{Q_1^2}{P_1} > 1-qC_y^2(1-\rho_{xy}^2)$$

- Comparing the proposed estimator t_s^m with the improved log estimator t_s, we get

$$min.MSE(t_s^m) < MSE(t_s) \implies \frac{Q_1^2}{P_1} > \frac{B^2}{A}$$

- Comparing the proposed estimator t_s^m with the memory-type ratio estimator t_r^m, we get

$$min.MSE(t_s^m) < MSE(t_r^m) \implies \frac{Q_1^2}{P_1} > 1-\zeta q\left(C_y^2+C_x^2-2\rho_{xy}C_xC_y\right)$$

- Comparing the proposed estimator t_s^m with the memory-type log estimator t_{bg}^m, we get

$$min.MSE(t_s^m) < MSE(t_{bg}^m) \implies \frac{Q_1^2}{P_1} > 1-\zeta qC_y^2(1-\rho_{xy}^2)$$

Under the aforesaid mathematical conditions, the developed estimator represses the reviewed estimators. This fact can only be checked through empirical study which is conducted in the next section.

5. Empirical study

In this section, the theoretical results are validated numerically using simulation and real data application.

5.1 Simulation

In the simulation, we assess the execution of the HEWMA-based memory-type log estimator under different conditions. These conditions may include varying levels of smoothing parameter λ_2, sample sizes n, and correlation coefficients ρ_{xy}. By systematically varying these parameters, we can assess the robustness, accuracy, and efficiency of the estimators across different scenarios. The algorithm of the simulation is delineated below.

(i). Use *R* software and artificially generate the following populations:
(a). Generate a normal (symmetric) population of size $N = 1000$ utilizing $\bar{X} = 10$, $\bar{Y} = 20$, $\sigma_x^2 = 25$, $\sigma_y^2 = 36$, and varying correlation coefficient $\rho_{xy} = 0.1, 0.3, 0.5, 0.7, 0.9$.
(b). Following Singh and Horn (1998) and Kumar and Siddiqui (2024), generate Chi-square (asymmetric) population of size N=1000 units with variables x and y through the following model:

$$Y = 10.6 + \sqrt{(1-\rho_{xy}^2)}\ Y^* + \rho_{xy}\left(\frac{S_y}{S_x}\right)X^*$$
$$X = 6.2 + X^*$$

where $x_i^* \sim \chi^2_{(11)}$ and $y_i^* \sim \chi^2_{(12)}$.

(ii). Draw several samples of sizes n = 15, 30, 60, 120, 240, 480 from the above generated populations and calculate the necessary statistics.

(iii). Consider 20,000 iterations and tabulate MSE of different estimators using (5.1) for above selected samples for varying values of ρ_{xy}=0.1, 0.3, 0.5, 0.7, 0.9 and smoothing constant λ_2 = 0.15, 0.55, 0.95 at fixed value of $\lambda_1 = 0.1$.

$$MSE(t) = \frac{1}{20,000}\sum_{i=1}^{20,000}(t_i - \bar{Y})^2 \tag{5.1}$$

where $t = t_r,\ t_{bg},\ t_s,\ t_r^m,\ t_{bg}^m,\ t_s^m$.

(iv). Report results by MSE in Tables 1-2 for normal and Chi-square (χ^2) populations, respectively.

From the results reported in Tables 1-2, it is noticed that the MSE of the conventional and memory-type estimators boils down as the amounts of ρ_{xy} vary from 0.1 to 0.9. For instance, from the results of Table 1, at $n = 15$, the MSE of the suggested estimator t_s^m is 0.072 and 0.014 for $\rho_{xy} = 0.1$ and $\rho_{xy} = 0.9$, respectively. Also, from the results of Table 2, at $n = 15$, the MSE of the suggested estimator t_s^m is 0.048 and 0.009 for $\rho_{xy} = 0.1$ and $\rho_{xy} = 0.9$, respectively.

The conventional and memory-type estimators' MSE boils down as the sample size increases for every value of correlation coefficient. For example, from the results of Table 1, at fixed $\rho_{xy} = 0.1$ and $\lambda_2 = 0.55$, the MSE of the proposed estimator t_s^m is 0.105 and 0.004 for $n = 15$ and $n = 480$, respectively. Also, from the results of Table 2, at fixed $\rho_{xy} = 0.1$ and $\lambda_2 = 0.55$, the MSE of the proposed estimator t_s^m is 0.071 and 0.002 for $n = 15$ and $n = 480$, respectively.

From the results of Tables 1-2, the MSE of the memory-type estimators increases for varying value of smoothing constant $\lambda_2 = 0.15$ to 0.95 at fixed value of $\lambda_1 = 0.1$. Here, the value of λ_1 is fixed because the parameter λ_2 directly governs the degree of smoothing in the short-term moving average (E_t), which in turn significantly influences the responsiveness of the HEWMA statistic to recent data variations. Therefore, varying λ_2 allows us to assess the estimator's sensitivity to short-term memory, which is critical in dynamic environments. The parameter λ_1, on the other hand, controls the overall memory depth (HE_t) and is held fixed to isolate the effects of λ_2 for clearer interpretation of the simulation outcomes.

Table 1. MSE of conventional and memory-type estimators for normal population when $\lambda_1 = 0.1$ and $\lambda_2 = (0.15, 0.55, 0.95)$

					$\lambda_2 = 0.15$			$\lambda_2 = 0.55$			$\lambda_2 = 0.95$		
ρ_{xy}	n	t_r	t_{bg}	t_s	t_r^m	t_{bg}^m	t_s^m	t_r^m	t_{bg}^m	t_s^m	t_r^m	t_{bg}^m	t_s^m
0.1	15	8.734	4.025	2.217	0.280	0.129	0.072	0.412	0.190	0.105	0.455	0.210	0.116
	30	4.291	2.098	1.161	0.138	0.067	0.037	0.202	0.099	0.055	0.224	0.109	0.061
	60	2.128	1.071	0.594	0.068	0.034	0.019	0.100	0.050	0.028	0.111	0.056	0.031
	120	1.060	0.541	0.300	0.034	0.017	0.010	0.050	0.025	0.014	0.055	0.028	0.016
	240	0.528	0.272	0.151	0.017	0.009	0.005	0.025	0.013	0.007	0.028	0.014	0.008
	480	0.264	0.136	0.076	0.008	0.004	0.002	0.012	0.006	0.004	0.014	0.007	0.004
0.3	15	6.824	3.807	2.089	0.219	0.122	0.068	0.322	0.179	0.1	0.356	0.198	0.110
	30	3.346	1.979	1.093	0.107	0.063	0.035	0.158	0.093	0.052	0.174	0.103	0.057
	60	1.658	1.008	0.559	0.053	0.032	0.018	0.078	0.048	0.026	0.086	0.053	0.029
	120	0.825	0.509	0.282	0.026	0.016	0.009	0.039	0.024	0.013	0.043	0.027	0.015
	240	0.411	0.256	0.142	0.013	0.008	0.005	0.019	0.012	0.007	0.021	0.013	0.007
	480	0.206	0.128	0.071	0.007	0.004	0.002	0.010	0.006	0.003	0.011	0.007	0.004
0.5	15	5.086	3.167	1.729	0.163	0.102	0.056	0.240	0.149	0.083	0.265	0.165	0.092
	30	2.492	1.642	0.905	0.080	0.053	0.029	0.117	0.077	0.043	0.130	0.086	0.048
	60	1.235	0.836	0.462	0.040	0.027	0.015	0.058	0.039	0.022	0.064	0.044	0.024
	120	0.614	0.421	0.234	0.020	0.014	0.008	0.029	0.020	0.011	0.032	0.022	0.012
	240	0.306	0.212	0.117	0.010	0.007	0.004	0.014	0.010	0.006	0.016	0.011	0.006
	480	0.153	0.106	0.059	0.005	0.003	0.002	0.007	0.005	0.003	0.008	0.006	0.003
0.7	15	3.417	2.150	1.165	0.110	0.069	0.038	0.161	0.101	0.056	0.178	0.112	0.062
	30	1.673	1.114	0.612	0.054	0.036	0.020	0.079	0.053	0.029	0.087	0.058	0.032
	60	0.829	0.567	0.313	0.027	0.018	0.010	0.039	0.027	0.015	0.043	0.030	0.016
	120	0.412	0.286	0.158	0.013	0.009	0.005	0.019	0.013	0.007	0.021	0.015	0.008
	240	0.206	0.143	0.080	0.007	0.005	0.003	0.010	0.007	0.004	0.011	0.007	0.004
	480	0.103	0.072	0.040	0.003	0.002	0.001	0.005	0.003	0.002	0.005	0.004	0.002
0.9	15	1.800	0.791	0.415	0.058	0.025	0.014	0.085	0.037	0.021	0.094	0.041	0.023
	30	0.879	0.411	0.222	0.028	0.013	0.007	0.041	0.019	0.011	0.046	0.021	0.012
	60	0.435	0.209	0.115	0.014	0.007	0.004	0.021	0.010	0.005	0.023	0.011	0.006
	120	0.217	0.105	0.058	0.007	0.003	0.002	0.010	0.005	0.003	0.011	0.005	0.003
	240	0.108	0.053	0.029	0.003	0.002	0.001	0.005	0.002	0.001	0.006	0.003	0.002
	480	0.054	0.027	0.015	0.002	0.001	0.000	0.003	0.001	0.001	0.003	0.001	0.001

Moreover, from the results of Tables 1-2, the proposed improved memory-type log estimator t_s^m dominate the traditional ratio estimator t_r, conventional log estimator t_{bg}, improved log estimator t_s, memory-type ratio estimator t_r^m, and memory-type log estimator t_{bg}^m for varying correlation coefficients, sample sizes, and smoothing constants.

Table 2. MSE of conventional and memory-type estimators for Chi-square population when $\lambda_1 = 0.1$ and $\lambda_2 = (0.15, 0.55, 0.95)$

					$\lambda_2 = 0.15$			$\lambda_2 = 0.55$			$\lambda_2 = 0.95$		
ρ_{xy}	n	t_r	t_{bg}	t_s	t_r^m	t_{bg}^m	t_s^m	t_r^m	t_{bg}^m	t_s^m	t_r^m	t_{bg}^m	t_s^m
0.1	15	4.030	2.702	1.495	0.129	0.087	0.048	0.190	0.127	0.071	0.210	0.141	0.078
	30	2.015	1.402	0.778	0.065	0.045	0.025	0.095	0.066	0.037	0.105	0.073	0.041
	60	1.007	0.714	0.396	0.032	0.023	0.013	0.047	0.034	0.019	0.052	0.037	0.021
	120	0.503	0.360	0.200	0.016	0.012	0.006	0.024	0.017	0.009	0.026	0.019	0.010
	240	0.252	0.181	0.100	0.008	0.006	0.003	0.012	0.009	0.005	0.013	0.009	0.005
	480	0.126	0.091	0.050	0.004	0.003	0.002	0.006	0.004	0.002	0.007	0.005	0.003
0.3	15	3.517	2.483	1.373	0.113	0.080	0.044	0.166	0.117	0.065	0.183	0.129	0.072
	30	1.758	1.289	0.715	0.056	0.041	0.023	0.083	0.061	0.034	0.092	0.067	0.037
	60	0.878	0.656	0.364	0.028	0.021	0.012	0.041	0.031	0.017	0.046	0.034	0.019
	120	0.439	0.331	0.184	0.014	0.011	0.006	0.021	0.016	0.009	0.023	0.017	0.010
	240	0.220	0.166	0.092	0.007	0.005	0.003	0.010	0.008	0.004	0.011	0.009	0.005
	480	0.110	0.083	0.046	0.004	0.003	0.001	0.005	0.004	0.002	0.006	0.004	0.002
0.5	15	2.802	2.047	1.131	0.090	0.066	0.036	0.132	0.096	0.054	0.146	0.107	0.059
	30	1.400	1.062	0.589	0.045	0.034	0.019	0.066	0.050	0.028	0.073	0.055	0.031
	60	0.699	0.541	0.300	0.022	0.017	0.010	0.033	0.025	0.014	0.036	0.028	0.016
	120	0.350	0.273	0.151	0.011	0.009	0.005	0.016	0.013	0.007	0.018	0.014	0.008
	240	0.175	0.137	0.076	0.006	0.004	0.002	0.008	0.006	0.004	0.009	0.007	0.004
	480	0.088	0.069	0.038	0.003	0.002	0.001	0.004	0.003	0.002	0.005	0.004	0.002
0.7	15	1.895	1.392	0.767	0.061	0.045	0.025	0.089	0.066	0.036	0.099	0.073	0.040
	30	0.947	0.722	0.400	0.030	0.023	0.013	0.045	0.034	0.019	0.049	0.038	0.021
	60	0.473	0.368	0.204	0.015	0.012	0.007	0.022	0.017	0.010	0.025	0.019	0.011
	120	0.236	0.185	0.103	0.008	0.006	0.003	0.011	0.009	0.005	0.012	0.010	0.005
	240	0.118	0.093	0.052	0.004	0.003	0.002	0.006	0.004	0.002	0.006	0.005	0.003
	480	0.059	0.047	0.026	0.002	0.001	0.001	0.003	0.002	0.001	0.003	0.002	0.001
0.9	15	0.793	0.519	0.283	0.025	0.017	0.009	0.037	0.024	0.014	0.041	0.027	0.015
	30	0.396	0.269	0.148	0.013	0.009	0.005	0.019	0.013	0.007	0.021	0.014	0.008
	60	0.198	0.137	0.076	0.006	0.004	0.002	0.009	0.006	0.004	0.010	0.007	0.004
	120	0.099	0.069	0.038	0.003	0.002	0.001	0.005	0.003	0.002	0.005	0.004	0.002
	240	0.050	0.035	0.019	0.002	0.001	0.001	0.002	0.002	0.001	0.003	0.002	0.001
	480	0.025	0.017	0.010	0.001	0.001	0.000	0.001	0.001	0.000	0.001	0.001	0.001

5.2 Real data application

In this section, an illustration of the proposed estimators is provided utilizing two real populations outlined below:

Population 1: Singh (2003), pages 1116-1118 is the source of this population. The data is consisting of the fish caught by marine recreational fishermen categorized by species group and year along with the Atlantic and Gulf coasts in the time period 1992 to 1995. In this population, “the amount of fish caught in 1995" denotes as the study variable, whereas “the amount of fish caught in 1994" denotes as an auxiliary variable. The density representing the auxiliary and study variables of this population are shown in Figure 1 and Figure 2, respectively. The characteristics of this population are described as: $N = 69$, $n = 30$, $\bar{X} = 4954.435$, $\bar{Y} = 4514.899$, $S_x^2 = 49829270$, $S_y^2 = 37199578$, and $\rho_{xy} = 0.9601$.

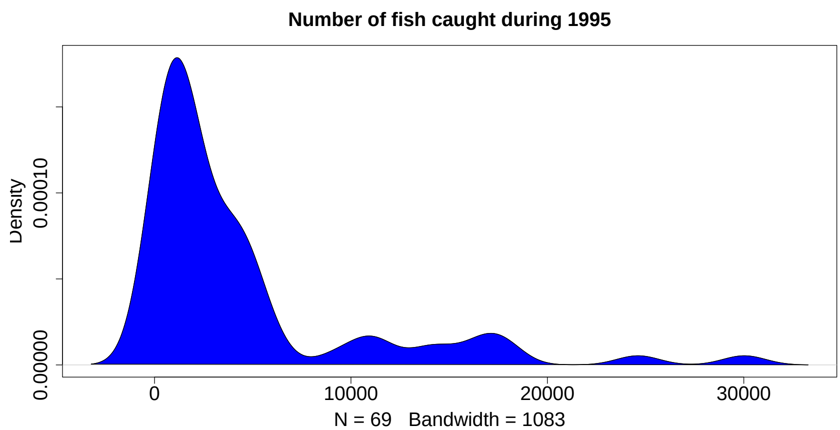

Figure 1. Density plot of study variable for population 1.

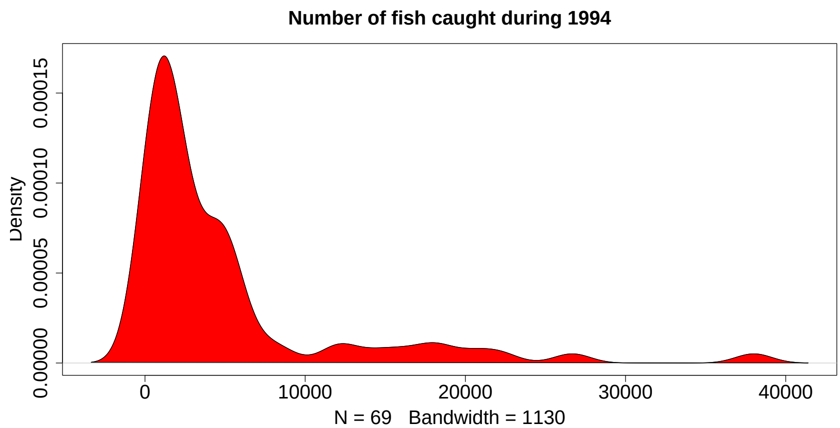

Figure 2. Density plot of auxiliary variable for population 1.

Population 2: The data is taken from Kadilar and Cingi (2003) which is consisting of the amount of apple production (taken as variable y) and number of apple trees (taken as variable x) in 106 villages of Marmara region in Turkey during the year 1999. The density plots of the study and auxiliary variables are provided in Figure 3 and Figure 4, respectively. The required parameters to compute the characteristics of different estimators are given as follows: $N = 106$, $\bar{X} = 24375.59$, $\bar{Y} = 1536.77$, $S_x = 49189.08$, $S_y = 6425.08$ and $\rho = 0.81$.

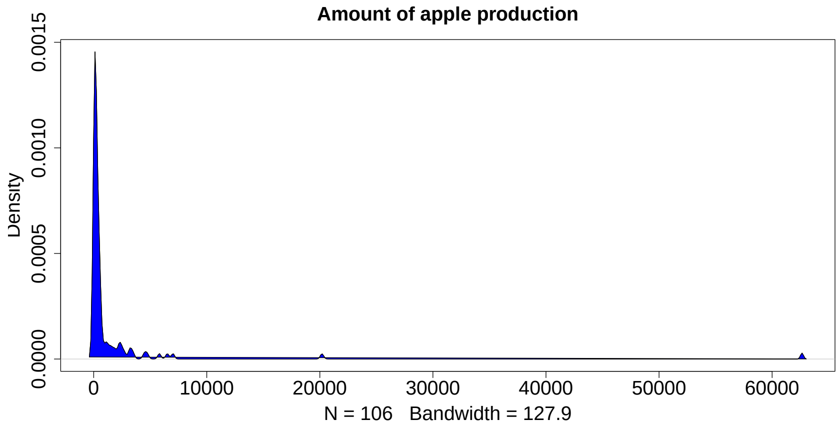

Figure 3. Density plot of study variable for population 2.

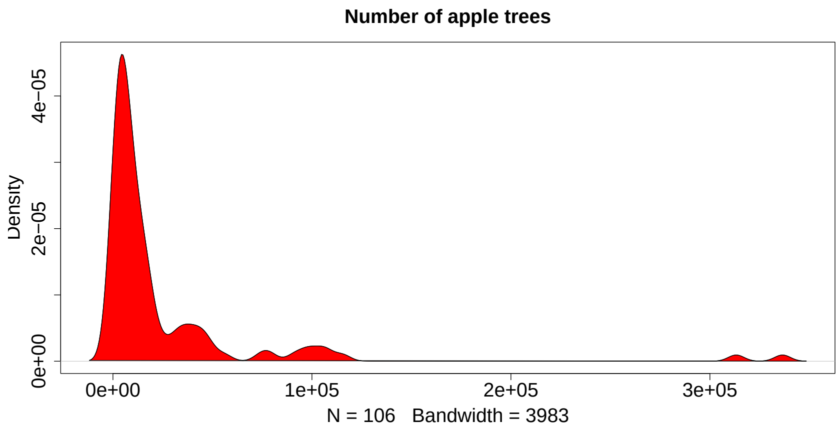

Figure 4. Density plot of auxiliary variable for population 2.

From the findings of Table 3, it can be observed that the MSE and RE of the proposed efficient memory-type log estimator are lesser and greater than the MSE and RE of the conventional ratio estimator t_r, log estimator t_{bg}, improved log estimator t_s, memory-type ratio estimator t_r^m, and memory-type log estimator t_{bg}^m in both the populations. Moreover, it is also noticed that the MSE and RE of the memory-type estimators decrease and increase as the value of smoothing parameter λ_2 increases in both the populations. For instance, at $\lambda_2 = 0.1$, the MSE of the proposed estimator

t_s^m is 1665.36, while, at $\lambda_2 = 0.9$, the MSE of estimator t_s^m is 2442.35.

Table 3. MSE and RE of the traditional and memory-type estimators for fixed value of $\lambda_1 = 0.05$ and varying values of $\lambda_2 = (0.1, 0.3, 0.5, 0.7, 0.9)$

		Population 1		Population 2	
λ_2	Estimators	MSE	RE	MSE	RE
	t_r	108072.60	11.47	735811.50	2.24
	t_{bg}	96976.81	12.78	552728.00	2.98
	t_s	69053.33	17.95	415661.20	3.97
0.1	t_r^m	1865.83	664.57	12703.53	129.98
	t_{bg}^m	1674.27	740.61	9542.65	173.04
	t_s^m	1665.36	744.57	9489.04	174.01
0.3	t_r^m	2430.48	510.18	16547.93	99.78
	t_{bg}^m	2180.94	568.55	12430.50	132.84
	t_s^m	2165.83	572.52	12339.69	133.81
0.5	t_r^m	2595.15	477.80	17669.06	93.45
	t_{bg}^m	2328.70	532.47	13272.67	124.41
	t_s^m	2311.47	536.44	13169.18	125.38
0.7	t_r^m	2681.65	462.39	18258.02	90.44
	t_{bg}^m	2406.32	515.30	13715.09	120.39
	t_s^m	2387.93	519.27	13604.62	121.37
0.9	t_r^m	2743.25	452.01	18677.44	88.40
	t_{bg}^m	2461.60	503.73	14030.15	117.69
	t_s^m	2442.35	507.70	13914.56	118.67

6. Conclusions

This work presents an effective memory-type log estimator utilising SRS based on the HEWMA approach. To the first order approximation, the MSE of the suggested estimator is produced. By contrasting the MSEs of the suggested and current conventional and memory-type estimators, the efficiency conditions were determined. We have shown the effectiveness and efficiency of the suggested estimator in a number of cases thorough testing and analysis, demonstrating its improved performance over current estimators. The suggested estimators are appropriate for real-world situations where memory-type estimation is crucial because they strike a compromise between computational cost and accuracy. In addition, the conclusions of the paper offer valuable perspectives on the fundamental techniques of log estimation and establish a foundation for further investigation in this field.

The suggested estimator has potential applications in finance, healthcare, economics, and environmental research, among other fields. For instance, in finance, it may help with more effective assessment of asset returns or market indexes, resulting in improved investment decision-making. A precise population mean estimate has the potential to improve epidemiological research in the healthcare industry, which might result in improved public health treatments and policies. Additionally, the suggested estimator may be applied in environmental sciences to estimate the population mean of ecological indicators or pollutant levels, supporting environmental management and monitoring procedures. It is thus recommended that survey practitioners use the suggested population mean estimation approach to their practical issues.

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Conflicts of Interest

The authors declare no conflict of interest.

Author Contributions

Conceptualization: KUMAR, A., Data curation: PARTIBHA, Formal analysis: PARTIBHA, SINGH, C., Funding acquisition: SINGH, C., Investigation: PARTIBHA, Methodology: KUMAR, A., Project administration: PARTIBHA, Software: PARTIBHA, Resources: KUMAR, A., PARTIBHA, Supervision: KUMAR, A., Validation: KUMAR, A., Visualization: KUMAR, A., PARTIBHA, Writing - original draft: PARTIBHA, Writing - review and editing: KUMAR, A., PARTIBHA, SINGH, C.

References

[1] Aslam, I., Noor-ul-Amin, M., Yasmeen, U., and Hanif, M. Memory type ratio and product estimators in stratified sampling. Journal of Reliability and Statistical Studies, 1-20 (2020). https://doi.org/10.13052/JRSS0974-8024.1311

[2] Aslam, I., Noor-ul-Amin, M., Hanif, M., and Sharma, P. Memory type ratio and product estimators under ranked-based sampling schemes. Communications in Statistics-Theory and Methods, **52** (4), 1155–1177 (2021). https://doi.org/10.1080/03610926.2021.1924784

[3] Bhushan, S. and Gupta, R. Some log-type classes of estimators using auxiliary information. International Journal of Agricultural and Statistical Sciences,**11**(2), 487-491 (2015). https://connectjournals.com/file_full_text/2415802H_487-491.pdf

[4] Bhushan, S., Kumar, A., Al-Omari, A.I., and Alomani, G.A. Mean estimation for time-based surveys using memory-type logarithmic estimators. Mathematics, **11**(9), 2125 (2023). https://doi.org/10.3390/math11092125

[5] Bhushan, S., Kumar, A., Alrumayh, A., Khogeer, H.A., and Onyango, R. Evaluating the performance of memory-type logarithmic estimators using simple random sampling. Plos one. **17**(12), e0278264 (2022) https://doi.org/10.1371/journal.pone.0278264

[6] Cochran, W.G. The estimation of the yields of cereal experiments by sampling for the ratio of grain to total produce. Journal of Agricultural Science, **30** (2), 262-275 (1940). doi:10.1017/S0021859600048012

[7] Kumar, A. and Siddiqui, A.S. Enhanced estimation of population mean using simple random sampling, Research in Statistics, **2** (1), 2335949 (2024). doi:10.1080/27684520.2024.2335949

[8] Haq, A. A new hybrid exponentially weighted moving average control chart for monitoring process mean. Quality and Reliability Engineering International. **29** (7), 1015-1025 (2013). https://doi.org/10.1002/qre.1453

[9] Haq, A. A new hybrid exponentially weighted moving average control chart for monitoring process mean: discussion. Quality and Reliability Engineering International, **33**(7), 1629-1631 (2016). https://doi.org/10.1002/qre.2092

[10] Kadilar, C. and Cingi, H. Ratio estimators in stratified random sampling. Biometric Journal, **45**, 218-225 (2003). https://doi.org/10.1007/s41872-018-0046-8

[11] Noor-ul-Amin, M. Memory type ratio and product estimators for population mean for time-based surveys. Journal of Statistical Computation and Simulation, **90** (17), 3080-3092 (2020). doi:10.1080/00949655.2020.1795660

[12] Noor-ul-Amin, M. Memory type estimators of population mean using exponentially weighted moving averages for time scaled surveys. Communications in Statistics-Theory and Methods, **50** (12), 2747-2758 (2019). https://doi.org/10.1080/03610926.2019.1670850

[13] Roberts, S. Control chart tests based on geometric moving averages. Technometrics, **1**(3), 239-250 (1959). https://doi.org/10.2307/1271439

[14] Qureshi, M.N., Tariq, M.U., and Hanif, M. Memory-type ratio and product estimators for population variance using exponentially weighted moving averages for time-scaled surveys. Communications in Statistics-Simulation and Computation, **53** (3), 1484-1493 (2022). https://doi.org/10.1080/03610918.2022.2050390

[15] Singh, S. Advanced sampling theory with applications: How Michael selected Amy, vol. 1&2. The Netherlands: Kluwer, (2003).https://link.springer.com/book/10.1007/978-94-007-0789-4

[16] Singh, H.P. and Horn, S. An alternative estimator for multi-character surveys. Metrika, **48**, 99-107 (1998). https://doi.org/10.1007/PL00020899

Appendix A

This section consider the proof of Z_t and Q_t as an unbiased estimators of $\bar{Y}$ and $\bar{X}$.

We consider the recursive definitions of the HEWMA statistics for the study and auxiliary variables:

$$E_{ty} = \lambda_2 \bar{y}_t + (1-\lambda_2) E_{ty-1}, \quad E_{ty0} = 0 \tag{A.1}$$
$$Z_t = \lambda_1 E_{ty} + (1-\lambda_1) Z_{t-1}, \quad Z_0 = 0 \tag{A.2}$$

$$E_{tx} = \lambda_2 \bar{x}_t + (1-\lambda_2) E_{tx-1}, \quad E_{tx0} = 0 \tag{A.3}$$
$$Q_t = \lambda_1 E_{tx} + (1-\lambda_1) Q_{t-1}, \quad Q_0 = 0 \tag{A.4}$$

Taking expectation on both sides of (A.1), we get

$$\mathbb{E}[E_{ty}] = \lambda_2 \mathbb{E}[\bar{y}_t] + (1-\lambda_2)\mathbb{E}[E_{ty-1}]$$

Since $\mathbb{E}[\bar{y}_t] = \bar{Y}$, we get

$$\mathbb{E}[E_{ty}] = \lambda_2 \bar{Y} + (1-\lambda_2)\mathbb{E}[E_{ty-1}]$$

Solving this recurrence with $E_{ty0} = 0$, we obtain

$$E[E_{ty}] = \bar{Y}(1-(1-\lambda_2)^t)$$

As $t \to \infty$, $(1-\lambda_2)^t \to 0$, so

$$\lim_{t\to\infty} \mathbb{E}[E_{ty}] = \bar{Y}$$

Taking expectation on both sides of (A.2), we get

$$\mathbb{E}[Z_t] = \lambda_1 \mathbb{E}[E_{ty}] + (1-\lambda_1)\mathbb{E}[Z_{t-1}]$$

Substitute $E[E_{ty}] = \bar{Y}(1-(1-\lambda_2)^t)$, then solve this recurrence with $Z_0 = 0$, to get

$$\mathbb{E}[Z_t] = \bar{Y}(1-(1-\lambda_1)^t)$$

Thus, $\lim_{t\to\infty} \mathbb{E}[Z_t] = \bar{Y}$, i.e., Z_t is asymptotically unbiased for $\bar{Y}$.

By symmetry, the same steps hold for the auxiliary variable as

$$\mathbb{E}[Q_t] = \bar{X}(1-(1-\lambda_1)^t) \Rightarrow \lim_{t\to\infty} \mathbb{E}[Q_t] = \bar{X}$$

Hence, the HEWMA-based estimators Z_t and Q_t are asymptotically unbiased estimators of the population means $\bar{Y}$ and $\bar{X}$, respectively:

$$\lim_{t\to\infty} \mathbb{E}[Z_t] = \bar{Y}, \quad \lim_{t\to\infty} \mathbb{E}[Q_t] = \bar{X}.$$