MAXIMUM LIKELIHOOD ESTIMATION FOR THE WEIGHTED LINDLEY DISTRIBUTION PARAMETERS UNDER DIFFERENT TYPES OF CENSORING

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▪ *ABSTRACT: In this paper the maximum likelihood estimators for the parameters of the weighted Lindley distribution considering different types of censoring, such as, type I, type II and random censoring are presented. A numerical simulation study is performed in order to verify the efficiency of our proposed methodology, which is also fully illustrated on two real data set.*

▪ *KEYWORDS: Weighted Lindley distribution; maximum likelihood estimation; censored data; random censoring.*

1 Introduction

Ghitany et al. (2008) investigated many properties of the Lindley distribution and outlined that such model provide better fit than the exponential distribution. From then on, many generalizations of the Lindley distribution have been introduced in the literature, such as the generalized Lindley (ZALERZADEH and DOLATI, 2009), weighted Lindley (GHITANY *et al.*, 2011), extended Lindley (BAKOUCH *et al.*, 2012), among others. Let T be a random variable representing a lifetime data with weighted Lindley (WL) distribution, the probability density function (p.d.f) is given by

$$f(t|\lambda,\phi) = \frac{\lambda^{\phi+1}}{(\lambda+\phi)\Gamma(\phi)} t^{\phi-1}(1+t)e^{-\lambda t}, \tag{1}$$

for all $t > 0$, $\phi > 0$ and $\lambda > 0$ and $\Gamma(\phi) = \int_0^\infty e^{-x}x^{\phi-1}dx$ known as gamma function. The hazard function can has an increasing or bathtub shape depending

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on the values of the parameters, in which provided a great flexibility for be used in different areas.

Ghitany *et al.* (2011) discussed many mathematical properties of the WL distribution and also the parameter estimation based on the maximum likelihood method (MLE). Mazucheli *et al.* (2013) compared the efficiency of four estimation methods: the MLE, method of moments, ordinary least-squares, and weighted least-squares and conclude that the weighted least-squares method reproduces similar results to those obtained using the maximum likelihood. Wang (2015) proposed a bias-corrected technique for the MLEs and argues that such procedure is highly recommended instead of estimators without bias-correction. Considering the Bayesian approach, Ali (2013) discussed different non-informative and informative priors for the parameters of the WL distribution. In the reliability context, Al-Mutairi *et al.* (2015) deals with the estimation of the stress-strength parameter $R = P(Y < X)$, when X and Y are two independent random variables with WL distribution.

In studies that involves temporal responses, it is common the presence of incomplete or partial data, widely known as censored data (LAWLESS, 2002). Such partial data even incomplete, provide important information about the unknown parameters of interesting and the omission could result in biased estimators. For the WL distribution the estimation procedures available in the literature are not capable to include censored data. Therefore, in this paper we discuss the maximum likelihood estimation considering different types of censoring, such as type I, type II and random censoring. Some referred papers regarding to the reliability applications of those types of censoring can be seen in Ghitany and Al-Awadhi (2002), Goodman *et al.* (2006), Joarder *et al.* (2011), Iliopoulos and Balakrishnan (2011), Arbuckle *et al.* (2014). The originality of this study comes from the fact that for the WL distribution, there has been no previous work considering those censoring mechanisms.

The paper is organized as follows. In Section 2, we review some properties of the weighted Lindley distribution. In Section 3 we presented the maximum likelihood estimators of the parameters of the WL distributions considering different censoring mechanism. In Section 4 we carry out a simulation study in order to verify our proposed methods. In Section 5 we illustrate our proposed methodology by considering two real lifetime data sets. Finally, some comments are presented in Section 6.

2 Weighted Lindley distribution

Let T be a random variable representing a lifetime data with weighted Lindley distribution then its p.d.f can be expressed as a two-component mixture

$$f(t|\lambda,\phi) = pf_1(t|\lambda,\phi) + (1-p)f_2(t|\lambda,\phi),$$

where $p = \lambda/(\lambda+\phi)$ and $f_j(t|\lambda,\phi)$ has p.d.f Gamma$(\phi+j-1,\lambda)$ distribution, for $j = 1, 2$. These results are useful in order to derive different mathematical properties for the WL distribution. The mean and variance of the WL distribution can be easily computed by

$$\mu = \frac{\phi(\lambda + \phi + 1)}{\lambda(\lambda + \phi)}, \quad \sigma^2 = \frac{(\phi + 1)(\lambda + \phi)^2 - \lambda^2}{\lambda^2(\lambda + \phi)^2}.$$

The survival function of T representing a probability of an observation not fail until a specified time t is

$$S(t|\lambda, \phi) = \frac{(\lambda + \phi)\Gamma(\phi, \lambda t) + (\lambda t)^{\phi}\, e^{-\lambda t}}{(\lambda + \phi)\Gamma(\phi)},$$

where $\Gamma(x, y) = \int_x^{\infty} w^{y-1} e^{-x} dw$ is the upper incomplete gamma.

The hazard function of T that quantify the instantaneous risk of failure at a given time is given by

$$h(t|\lambda, \phi) = \frac{f(t|\lambda, \phi)}{S(t|\lambda, \phi)} = \frac{\lambda^{\phi+1} t^{\phi-1}(1 + t)e^{-\lambda t}}{(\lambda + \phi)\Gamma(\phi, \lambda t) + (\lambda t)^{\phi}\, e^{-\lambda t}}. \quad (2)$$

Ali (2013) obtain the Bonferroni and the Lorenz curves, various entropies and order statistics for the WL distribution. Ghitany *et al.* (2011) present some structural properties of the p.d.f, hazard function and the mean residual life function of the WL distribution. Moreover, Ghitany *et al.* (2011) proved that the hazard rate function (2) is bathtub (increasing) shaped if $0 < \phi < 1$ ($\phi \geq 1$) for all $\lambda > 0$. Figure 1 gives examples from the shapes of the hazard function for different values of ϕ and λ.

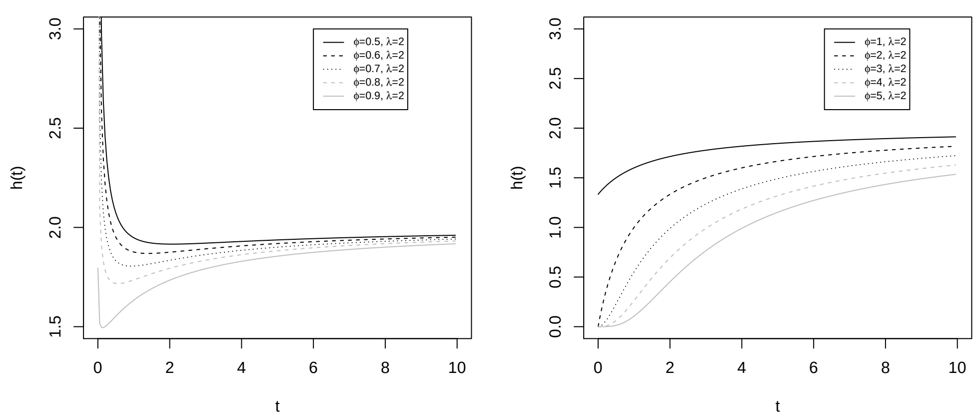

Figure 1 - Hazard rate for WL distribution considering different values of ϕ and λ.

3 Maximum likelihood estimation

Among the statistical inference methods, the maximum likelihood method is widely used due to its better asymptotic properties. We discuss the maximum likelihood estimator for the two parameters of the weighted Lindley distribution considering different types of censoring, such as type II, type I and random censoring. Other mechanisms of censoring as the progressive type II censoring (BALAKRISHNAN and AGGARWALA, 2000) and Hybrid censoring mechanism (BALAKRISHNAN and KUNDU, 2013) could also be obtained to WL distribution.

3.1 Type II censoring

In industrial experiments, the study of the lifetime of electronic components are usually finished after a fixed number of failures r, in this case $n-r$ will be the number of censored components. This mechanism of censoring is known as type II censoring (for more details see LAWLESS, 2002) and its likelihood function is given by

$$L(\lambda,\phi|\boldsymbol{t}) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} f(t_{(i)}|\lambda,\phi)S(t_{(r)}|\lambda,\phi)^{n-r},$$

where $t_{(i)}$ denotes the ith order statistic. Let $T_1,\cdots,T_n$ be a random sample with WL distribution the likelihood function considering type II censoring is given by

$$\begin{aligned} L(\lambda,\phi|\boldsymbol{t}) =& \frac{n!}{(n-r)!}\frac{\lambda^{r(\phi+1)}\left((\lambda+\phi)\Gamma(\phi,\lambda t_{(r)}) + \left(\lambda t_{(r)}\right)^{\phi} e^{-\lambda t_{(r)}}\right)^{n-r}}{(\lambda+\phi)^n\Gamma(\phi)^n}\times \\ &\times\prod_{i=1}^{r} t_{(i)}^{\phi-1}(1+t_{(i)})e^{-\lambda t_i}. \end{aligned} \tag{3}$$

The logarithm of the likelihood function (3) without the constant term is

$$\begin{aligned} l(\lambda,\phi|\boldsymbol{t}) =& -\lambda\sum_{i=1}^{r} t_i + (\phi-1)\sum_{i=1}^{r}\log(t_i) - n\log\left(\Gamma(\phi)\right) + r(\phi+1)\log(\lambda) \\ & - n\log(\phi+\lambda) + (n-r)\log\left((\lambda+\phi)\Gamma(\phi,\lambda t_{(r)}) + (\lambda t_{(r)})^{\phi}e^{-\lambda t_{(r)}}\right). \end{aligned} \tag{4}$$

From $\partial l(\lambda,\phi|\boldsymbol{t})/\partial\lambda = 0$ and $\partial l(\lambda,\phi|\boldsymbol{t})/\partial\phi = 0$ we obtain the likelihood equations respectively given by

$$\frac{n}{\lambda+\phi} - \frac{r(\phi+1)}{\lambda} + \sum_{i=1}^{r} t_i = \frac{(n-r)\left(\Gamma(\phi,\lambda t_{(r)})e^{-\lambda t_{(r)}} - (t_{(r)}+1)\left(\lambda t_{(r)}\right)^{\phi}\right)}{\left((\lambda+\phi)\Gamma(\phi,\lambda t_{(r)})e^{-\lambda t_{(r)}}\right) + \left(\lambda t_{(r)}\right)^{\phi}}, \tag{5}$$

$$\frac{(n-r)\left(\Gamma(\phi,\lambda t_{(r)}) + (\lambda+\phi)\Psi(\phi,\lambda t_{(r)}) + \left(\lambda t_{(r)}\right)^{\phi}\log(\lambda t_{(r)})e^{-\lambda t_{(r)}}\right)}{\left((\lambda+\phi)\Gamma(\phi,\lambda t_{(r)})\right) + \left(\lambda t_{(r)}\right)^{\phi} e^{-\lambda t_{(r)}}} = -r\log(\lambda)$$

$$+\frac{n}{\lambda+\phi}+n\psi(\phi)-\sum_{i=1}^{r}\log(t_i), \tag{6}$$

where $\Psi(k,x) = \int_x^\infty w^{k-1}\log(w)e^{-w}dw$. Numerical methods such as Newton-Raphson are required to find the solution of these non-linear system.

3.2 Type I censoring

Consider that n patients is in a treatment and suppose that $d < n$ has died before t_c, then $n-d$ patients are alive and will be censored. The likelihood function in this case is given by

$$L(\boldsymbol{\theta}, \boldsymbol{t}) = \prod_{i=1}^{n} f(t_i|\boldsymbol{\theta})^{\delta_i} S(t_c|\boldsymbol{\theta})^{n-d},$$

where $\delta_i = I(t_i \leq t_c)$ is an indicator function and $d = \sum_i^n \delta_i$ is a random variable. Let $T_1, \cdots, T_n$ be a random sample with WL distribution, in this case the likelihood function is given by

$$L(\lambda,\phi|\boldsymbol{t}) = \frac{\lambda^{d(\phi+1)}\left((\lambda+\phi)\Gamma(\phi,\lambda t_c)+(\lambda t_c)^{\phi}e^{-\lambda t_c}\right)^{n-d}}{(\lambda+\phi)^n\Gamma(\phi)^n}\times$$
$$\times\prod_{i=1}^{n}\left(t_i^{\phi-1}(1+t_i)e^{-\lambda t_i}\right)^{\delta_i}. \tag{7}$$

The logarithm of the likelihood function (7) without the constant term is given by

$$\begin{aligned} l(\lambda,\phi|\boldsymbol{t}) = &+(\phi-1)\sum_{i=1}^{n}\delta_i\log(t_i)-\lambda\sum_{i=1}^{n}\delta_i t_i+d(\phi+1)\log(\lambda)-n\log(\phi+\lambda)\\ &+(n-d)\log\left((\lambda+\phi)\Gamma(\phi,\lambda t_c)+(\lambda t_c)^{\phi}e^{-\lambda t_c}\right)-n\log\left(\Gamma(\phi)\right). \end{aligned} \tag{8}$$

From $\partial l(\lambda,\phi|\boldsymbol{t})/\partial\lambda = 0$ and $\partial l(\lambda,\phi|\boldsymbol{t})/\partial\phi = 0$, the likelihood equations are

$$\frac{n}{\lambda+\phi}-\frac{d(\phi+1)}{\lambda}+\sum_{i=1}^{n}\delta_i t_i = \frac{(n-d)\left(\Gamma(\phi,\lambda t_c)e^{-\lambda t_c}-(t_c+1)(\lambda t_c)^{\phi}\right)}{((\lambda+\phi)\Gamma(\phi,\lambda t_c)e^{-\lambda t_c})+(\lambda t_c)^{\phi}},$$

$$\frac{(n-d)\left(\Gamma(\phi,\lambda t_c)+(\lambda+\phi)\Psi(\phi,\lambda t_c)+(\lambda t_c)^{\phi}\log(\lambda t_c)e^{-\lambda t_c}\right)}{((\lambda+\phi)\Gamma(\phi,\lambda t_c))+(\lambda t_c)^{\phi}e^{-\lambda t_c}} = -d\log(\lambda)$$
$$+\frac{n}{\lambda+\phi}+n\psi(\phi)-\sum_{i=1}^{n}\delta_i\log(t_i).$$

Numerical methods such as Newton-Raphson are required to find the solution of these non-linear system.

3.3 Random censoring

In medical survival analysis and industrial life testing, random censoring schemes has been receive special attention. Suppose that the ith individual has a lifetime T_i and a censoring time C_i, moreover the random censoring times C_is are independent of T_is and that their distribution does not depend on the parameters, then the data set is (t_i, δ_i), where $t_i = \min(T_i, C_i)$ and $\delta_i = I(T_i \leq C_i)$. This type of censoring have as special case the type I and II censoring mechanism. The likelihood function for $\boldsymbol{\theta}$ is given by

$$L(\boldsymbol{\theta}, \boldsymbol{t}) = \prod_{i=1}^{n} f(t_i|\boldsymbol{\theta})^{\delta_i} S(t_i|\boldsymbol{\theta})^{1-\delta_i}.$$

Let $T_1, \cdots, T_n$ be a random sample of WL distribution, the likelihood function considering data with random censoring is given by,

$$\begin{aligned} L(\lambda, \phi|\boldsymbol{t}) =& \frac{\lambda^{d(\phi+1)}}{(\lambda+\phi)^n \Gamma(\phi)^n} \prod_{i=1}^{n} \left((\lambda+\phi)\Gamma(\phi, \lambda t_i) + (\lambda t_i)^{\phi} e^{-\lambda t_i} \right)^{1-\delta_i} \times \\ & \times \left(t_i^{\phi-1} (1+t_i) e^{-\lambda t_i} \right)^{\delta_i}. \end{aligned} \tag{9}$$

The logarithm of the likelihood function (9) without the constant term is given by,

$$\begin{aligned} l(\lambda, \phi|\boldsymbol{t}) =& (\phi-1)\sum_{i=1}^{n} \delta_i \log(t_i) - \lambda \sum_{i=1}^{n} \delta_i t_i + d(\phi+1)\log(\lambda) - n\log(\phi+\lambda) \\ & + \sum_{i=1}^{n} (1-\delta_i) \log\left((\lambda+\phi)\Gamma(\phi, \lambda t_i) + (\lambda t_i)^{\phi} e^{-\lambda t_i} \right) - n\log\left(\Gamma(\phi)\right). \end{aligned} \tag{10}$$

From $\partial l(\lambda, c|\boldsymbol{t})/\partial\lambda = 0$ and $\partial l(\lambda, \phi|\boldsymbol{t})/\partial\phi = 0$, we get the likelihood equations,

$$\frac{n}{\lambda+\phi} - \frac{d(\phi+1)}{\lambda} + \sum_{i=1}^{n} \delta_i t_i = \sum_{i=1}^{n} \frac{(1-\delta_i)\left(\Gamma(\phi, \lambda t_i) e^{-\lambda t_i} - (t_i+1)(\lambda t_i)^{\phi} \right)}{((\lambda+\phi)\Gamma(\phi, \lambda t_i) e^{-\lambda t_i}) + (\lambda t_i)^{\phi}}, \tag{11}$$

$$\begin{aligned} \sum_{i=1}^{n} \frac{(1-\delta_i)\left(\Gamma(\phi, \lambda t_i) + (\lambda+\phi)\Psi(\phi, \lambda t_i) + (\lambda t_i)^{\phi} \log(\lambda t_i) e^{-\lambda t_i} \right)}{((\lambda+\phi)\Gamma(\phi, \lambda t_i)) + (\lambda t_i)^{\phi} e^{-\lambda t_i}} &= -d\log(\lambda) \\ + \frac{n}{\lambda+\phi} + n\psi(\phi) - \sum_{i=1}^{n} \delta_i \log(t_i) & \end{aligned} \tag{12}$$

Numerical methods are also required to find the solution of these non-linear equations.

4 Simulation study

In this section we present a simulation study via Monte Carlo methods. The main goal of these simulations is to study the efficiency of methodology. The results were computed using the software R (R CORE TEAM, 2016). The following procedure was adopted:

1. Generate values of the WL(ϕ, λ) with size n;
2. Using the values obtained in step 1, calculate the MLE $\hat{\phi}$ e $\hat{\lambda}$;
3. Repeat the steps 1 and 2 N times;
4. Using $\hat{\boldsymbol{\theta}} = (\hat{\phi}, \hat{\lambda})$ and $\boldsymbol{\theta} = (\phi, \lambda)$, compute the mean relative estimates (MRE) $\sum_{j=1}^{N} \frac{\hat{\theta}_{i,j}/\theta_i}{N}$, the mean square errors (MSE) $\sum_{j=1}^{N} \frac{(\hat{\theta}_{i,j}-\theta_i)^2}{N}$, the bias $\sum_{,=1}^{N} \frac{\hat{\theta}_{i,j}}{N} - \theta_i$ for $i = 1, 2$ and the 95% coverage probability.

Considering this approach it is expected that the Bias and MSE return values closer to zero and the MREs closer to one. The 95% coverage probabilities were also computed considering the 95% confidence interval. For a large number of experiments using a 95% confidence intervals, the frequencies of these intervals that covered the true values of $\boldsymbol{\theta}$ should be closer to 0.95. The coverage probabilities were calculated using the numeric observed information matrix obtained from the maxLik package results.

The seed used in the pseudo-random number generators was 2014. We fixed $N = 100,000$, $n = (5, 10, 25, 50, 100)$ and $\boldsymbol{\theta} = ((0.5, 2), (3, 2))$ with 20% and 40% of censored data, these values of $\boldsymbol{\theta}$ were selected in order to obtain the two possible hazard rate functions (increasing and bathtub shape). Moreover, we drawn the type II censored data setting the first r values as complete data and $n - r$ were censored in $t_{(r)}$. To generate type I and random censored data, we considered respectively the same procedures used by Goodman et. al. (2006) and Bayoud (2012). Considering these approaches it is expected that the mean for the proportions of the censored data ($E[p]$) will be approximately 0.2 and 0.4, where $p_j, j = 1, \ldots, N$ are proportions of the censored data for each sample. The maximum likelihood estimates were computed using the log-likelihood functions (4), (8) and (10) with the routine maxLik available in R to maximize such functions in which was able to locate the maximum for a wide range of initial values. The solution for the maximum was unique for all initial values.

Tables 1-3 displays the Mean, MREs, MSEs, Bias and the coverage probability ($C_{95\%}$) with a 95% confidence level for the MLEs considering N simulated samples, different values of n, 20% and 40% of censoring.

Some of the points are quite clear from the tables. The Bias and MSE for all parameters tend to zero as n increases, i.e. all estimators are consistent for the parameters and also as expected the values of MREs tend to one, i.e. the estimators are asymptotically unbiased for the parameters. Moreover, the coverage

Table 1 - Mean, MRE, MSE, Bias and C estimates for N samples of sizes $n = (5, 10, 25, 50, 100)$, with 20% and 40% of type II censored data

		Mean	MRE	MSE	Bias	$C_{95\%}$	Mean	MRE	MSE	Bias	$C_{95\%}$
n	r			$\phi = 0.5$					$\lambda = 2$		
5	4	0.922	1.845	0.810	0.422	0.969	4.764	2.382	30.240	2.764	0.961
10	8	0.699	1.399	0.199	0.199	0.971	3.354	1.677	8.414	1.354	0.965
25	20	0.565	1.131	0.033	0.065	0.963	2.435	1.217	1.243	0.435	0.962
50	40	0.530	1.061	0.012	0.030	0.957	2.198	1.099	0.393	0.198	0.957
100	80	0.515	1.029	0.005	0.015	0.953	2.095	1.048	0.157	0.095	0.954
5	3	0.975	1.950	0.990	0.475	0.968	5.840	2.920	55.550	3.840	0.950
10	6	0.750	1.501	0.281	0.250	0.971	4.162	2.081	19.370	2.162	0.959
25	15	0.589	1.177	0.049	0.089	0.968	2.767	1.383	3.220	0.767	0.962
50	30	0.540	1.081	0.016	0.040	0.960	2.335	1.168	0.867	0.335	0.960
100	60	0.519	1.039	0.006	0.019	0.953	2.158	1.079	0.309	0.158	0.956
n	r			$\phi = 3$					$\lambda = 2$		
5	4	4.385	1.462	10.280	1.385	0.950	2.978	1.489	4.874	0.978	0.955
10	8	4.007	1.336	5.116	1.007	0.959	2.678	1.339	2.243	0.678	0.961
25	20	3.514	1.171	1.802	0.514	0.959	2.338	1.169	0.753	0.338	0.961
50	40	3.265	1.088	0.740	0.265	0.958	2.172	1.086	0.303	0.172	0.958
100	80	3.129	1.043	0.304	0.129	0.955	2.084	1.042	0.125	0.084	0.955
5	3	4.395	1.465	12.250	1.395	0.944	3.108	1.554	7.022	1.108	0.947
10	6	4.050	1.350	5.994	1.050	0.954	2.785	1.393	3.159	0.785	0.956
25	15	3.623	1.208	2.342	0.623	0.960	2.447	1.224	1.158	0.447	0.961
50	30	3.351	1.117	1.048	0.351	0.958	2.247	1.124	0.499	0.247	0.959
100	60	3.175	1.058	0.428	0.175	0.956	2.123	1.062	0.201	0.123	0.957

probabilities of the parameters tend to 0.95 as n increase. Based on these results and through the maximum likelihood method we achieved good inferences for the parameters of the WL distribution considering different types of censoring.

Table 2 - Mean, MRE, MSE, Bias, C and $E[p]$ estimates for N samples of sizes $n = (5, 10, 25, 50, 100)$, with 20% and 40% of type I censored data

	Mean	MRE	MSE	Bias	$C_{95\%}$	Mean	MRE	MSE	Bias	$C_{95\%}$	$E[p]$
n			$\phi = 0.5$					$\lambda = 2$			
5	0.870	1.740	0.844	0.370	0.950	4.168	2.084	25.800	2.168	0.926	0.202
10	0.651	1.301	0.159	0.151	0.961	2.841	1.420	4.533	0.841	0.946	0.199
25	0.549	1.097	0.028	0.049	0.958	2.269	1.134	0.799	0.269	0.949	0.199
50	0.523	1.046	0.011	0.023	0.955	2.127	1.063	0.313	0.127	0.949	0.199
100	0.511	1.022	0.005	0.011	0.952	2.062	1.031	0.139	0.062	0.951	0.199
5	0.796	1.592	0.536	0.296	0.947	4.314	2.157	27.270	2.314	0.909	0.390
10	0.668	1.336	0.205	0.168	0.957	3.111	1.556	7.256	1.111	0.929	0.400
25	0.558	1.115	0.039	0.058	0.959	2.380	1.190	1.478	0.380	0.941	0.401
50	0.527	1.053	0.014	0.027	0.956	2.177	1.089	0.563	0.177	0.944	0.401
100	0.513	1.026	0.006	0.013	0.952	2.087	1.043	0.247	0.087	0.948	0.401
n			$\phi = 2$					$\lambda = 3$			
5	3.935	1.645	25.380	1.935	0.903	4.315	1.658	11.720	1.315	0.902	0.209
10	2.922	1.307	5.907	0.922	0.941	3.591	1.296	2.455	0.591	0.941	0.205
25	2.406	1.135	1.593	0.406	0.954	3.251	1.126	0.623	0.251	0.954	0.201
50	2.205	1.068	0.663	0.205	0.954	3.127	1.063	0.261	0.127	0.954	0.201
100	2.101	1.034	0.287	0.101	0.953	3.063	1.031	0.115	0.063	0.952	0.201
5	3.413	1.471	17.960	1.413	0.906	4.053	1.526	9.547	1.053	0.903	0.404
10	2.835	1.278	5.844	0.835	0.937	3.566	1.283	2.581	0.566	0.935	0.405
25	2.458	1.153	2.076	0.458	0.951	3.300	1.150	0.908	0.300	0.949	0.401
50	2.250	1.083	0.914	0.250	0.954	3.162	1.081	0.403	0.162	0.953	0.400
100	2.125	1.042	0.390	0.125	0.953	3.082	1.041	0.176	0.082	0.952	0.400

Table 3 - Mean, MRE, MSE, Bias, C and $E[p]$ estimates for N samples of sizes $n = (5, 10, 25, 50, 100)$, with 20% and 40% of random censored data

	Mean	MRE	MSE	Bias	$C_{95\%}$	Mean	MRE	MSE	Bias	$C_{95\%}$	$E[p]$
n			$\phi = 0.5$					$\lambda = 2$			
5	0.943	1.887	1.436	0.443	0.951	4.564	2.282	32.600	2.564	0.931	0.200
10	0.665	1.330	0.188	0.165	0.961	2.980	1.490	5.651	0.980	0.951	0.201
25	0.552	1.104	0.028	0.052	0.958	2.304	1.152	0.862	0.304	0.953	0.201
50	0.524	1.048	0.011	0.024	0.955	2.141	1.070	0.317	0.141	0.952	0.201
100	0.512	1.024	0.005	0.012	0.952	2.067	1.033	0.136	0.067	0.951	0.201
5	0.934	1.868	1.431	0.434	0.940	4.984	2.492	45.140	2.984	0.904	0.409
10	0.690	1.381	0.295	0.190	0.955	3.271	1.636	10.050	1.271	0.927	0.418
25	0.558	1.117	0.037	0.058	0.959	2.397	1.198	1.529	0.397	0.943	0.419
50	0.527	1.053	0.013	0.027	0.956	2.182	1.091	0.554	0.182	0.947	0.419
100	0.513	1.025	0.006	0.013	0.953	2.085	1.042	0.239	0.085	0.949	0.419
n			$\phi = 2$					$\lambda = 3$			
5	4.880	1.960	43.730	2.880	0.885	4.875	1.938	18.630	1.875	0.890	0.193
10	3.322	1.441	10.330	1.322	0.936	3.837	1.419	4.027	0.837	0.937	0.202
25	2.486	1.162	1.813	0.486	0.955	3.305	1.153	0.708	0.305	0.957	0.203
50	2.231	1.077	0.657	0.231	0.956	3.146	1.073	0.259	0.146	0.956	0.203
100	2.111	1.037	0.274	0.111	0.952	3.070	1.035	0.109	0.070	0.953	0.203
5	5.119	2.040	55.450	3.119	0.861	5.171	2.086	26.760	2.171	0.862	0.382
10	3.580	1.527	16.540	1.580	0.916	4.040	1.520	6.972	1.040	0.916	0.404
25	2.568	1.189	2.501	0.568	0.951	3.368	1.184	1.053	0.368	0.952	0.407
50	2.273	1.091	0.869	0.273	0.956	3.178	1.089	0.373	0.178	0.955	0.407
100	2.130	1.043	0.353	0.130	0.953	3.084	1.042	0.153	0.084	0.952	0.407

5 Application

In this section, we illustrated our proposed methodology by considering two data set. The results obtained from the WL distribution were compared with the Weibull, Gamma, Lognormal, and Logistic distributions and the nonparametric survival curve adjusted through the Kaplan-Meier estimator (KAPLAN and MEIER, 1958).

Initially, in order to verify the behavior of the empirical hazard function it will be considered the TTT-plot (total time on test) proposed by Barlow and Campo (1975). The TTT-plot is achieved through the consecutive plot of the values $[r/n, G(r/n)]$ where

$$G(r/n) = \left(\sum_{i=1}^{r} t_i + (n-r)t_{(r)}\right) / \sum_{i=1}^{n} t_i, \quad r = 1, \ldots, n, \; i = 1, \ldots, n \tag{13}$$

and t_i is the order statistics. If the curve is concave (convex), the hazard function is increasing (decreasing), when it starts convex and then concave (concave and then convex) the hazard function will have a bathtub (inverse bathtub) shape.

Different discrimination criterion methods based on log-likelihood function evaluated at the MLEs were also considered. Let k be the number of parameters to be fitted and $\hat{\boldsymbol{\theta}}$ the MLEs of $\boldsymbol{\theta}$, the discrimination criterion methods are, respectively, the Akaike information criterion (AIC) computed through $AIC = -2l(\hat{\boldsymbol{\theta}}; \boldsymbol{x}) + 2k$, Corrected Akaike information criterion $AICC = AIC + \frac{2\,k\,(k+1)}{(n-k-1)}$, Hannan-Quinn information criterion $HQIC = -2\,l(\hat{\boldsymbol{\theta}}; \boldsymbol{x}) + 2\,k\,\log(\log(n))$ and the consistent Akaike information criterion $CAIC = AIC + k\log(n) - k$. The best model is the one which provides the minimum values of those criteria.

5.1 Rats with vaginal cancer

Presented by Pike (1966) the data set represent the lifetimes of 40 rats with vaginal cancer exposed to the carcinogen DMBA. In Table 4, we reproduce the data related to the survival times (in days) of 40 rats (+ indicates the presence of censorship).

Table 4 - Dataset related to the lifetimes of 40 rats with vaginal cancer exposed to the carcinogen DMBA

143	164	188	188	190	192	206	209	213	216
220	227	230	234	246	265	304+	216+	244	142
156	173	198	205	232	232	233	233	233	233
239	240	261	280	280	296	296	323	204+	344+

Based on Table 4, the data clearly has random censoring mechanism, consequently the equations (11) and (12) were used to compute the MLEs. Table 5 displays the MLEs, standard-error and 95% confidence intervals for ϕ and λ. Table 6 presents the results of AIC, AICc, HQIC and the CAIC criteria, for different probability distributions.

Table 5 - MLE, Standard-error and 95% confidence intervals for ϕ and λ

$\boldsymbol{\theta}$	MLE	S.E	$CI_{95\%}(\theta)$
ϕ	21.7545	1.3254	(19.1566; 24.3523)
λ	0.0978	0.0066	(0.0848; 0.1109)

Table 6 - Results of AIC, AICc, HQIC, CAIC criteria for different probability distributions considering the dataset related to rats with vaginal cancer exposed to the carcinogen DMBA

Test	W. Lindley	Weibull	Gamma	Lognormal	Logistic
AIC	**390.342**	394.423	390.648	390.361	391.352
AICc	**386.666**	390.748	386.972	386.686	387.676
HQIC	**391.563**	395.645	391.869	391.583	392.573
CAIC	**395.719**	399.801	396.026	395.739	396.730

In the Figure 2, we have the TTT-plot, survival function adjusted by different distributions and Kaplan–Meier estimator.

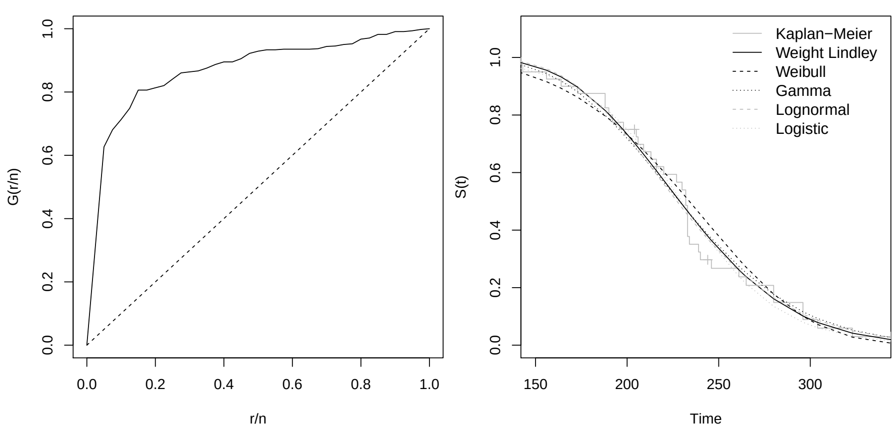

Figure 2 - TTT-plot, survival function adjusted by different distributions and the Kaplan–Meier estimator considering the lifetimes of 40 rats with cancer.

Based on the TTT-plot there is an indication that the hazard function has increasing failure rate. Comparing the empirical survival function with the adjusted models we observed a goodness of the fit for the weighted Lindley distribution. This result is also confirmed from the different discrimination criterion methods considered since WL distribution has the minimum value. Since $\hat{\phi} = 21.7545$ (increasing shape, $\phi > 1$), then the hazard function is increasing, confirming the result obtained from TTT-plot. Therefore, through our proposed methodology the data related to rats with vaginal cancer can be described by the weighted Lindley distribution.

5.2 Lifetime of electrical devises

Presented by Lawless (2002, p.112) the data set is related to 60 electrical devices. The survival times is given in cycles to failure divided by 1000 and was firstly presented without censoring. We considered that the experiment was finished after $r = 49$ failure, therefore $n - r = 11$ components were censored. Table 7 reproduces the lifetimes from the first 49 electrical devices.

Table 7 - Dataset related to the lifetimes of 60 (in cycles) electrical devices

0.014	0.034	0.059	0.061	0.069	0.080	0.123	0.142	0.165
0.210	0.381	0.464	0.479	0.556	0.574	0.839	0.917	0.969
0.991	1.064	1.088	1.091	1.174	1.270	1.275	1.355	1.397
1.477	1.578	1.649	1.702	1.893	1.932	2.001	2.161	2.292
2.326	2.337	2.628	2.785	2.811	2.886	2.993	3.122	3.248
3.715	3.79	3.857	3.912					

Since the experiment was finished after a predetermined number of failures r, then the data has type II censoring mechanism and the equations (5) and (6) were used to compute the MLEs. Table 8 displays the MLEs, standard-error and 95% confidence intervals for ϕ and λ. Table 9 presents the results of AIC, AICc, HQIC and the CAIC considering different probability distributions for the electrical data.

Table 8 - MLE, Standard-error and 95% confidence intervals for ϕ and λ considering the electrical devices data

$\boldsymbol{\theta}$	MLE	S.E	$CI_{95\%}(\theta)$
ϕ	0.6764	0.1341	(0.4137; 0.9392)
λ	0.5260	0.0954	(0.3391; 0.7129)

In the Figure 3, we have the TTT-plot and the survival function adjusted by different distributions and the Kaplan-Meier estimator.

Table 9 - Results of AIC, AICc, HQIC, CAIC considering different probability distributions for the electrical data

Test	W. Lindley	Weibull	Gamma	Lognormal	Logistic
AIC	**185.174**	186.596	186.218	195.022	220.103
AICc	**181.384**	182.807	182.428	191.233	216.314
HQIC	**186.812**	188.235	187.856	196.660	221.742
CAIC	**191.363**	192.785	192.407	201.211	226.292

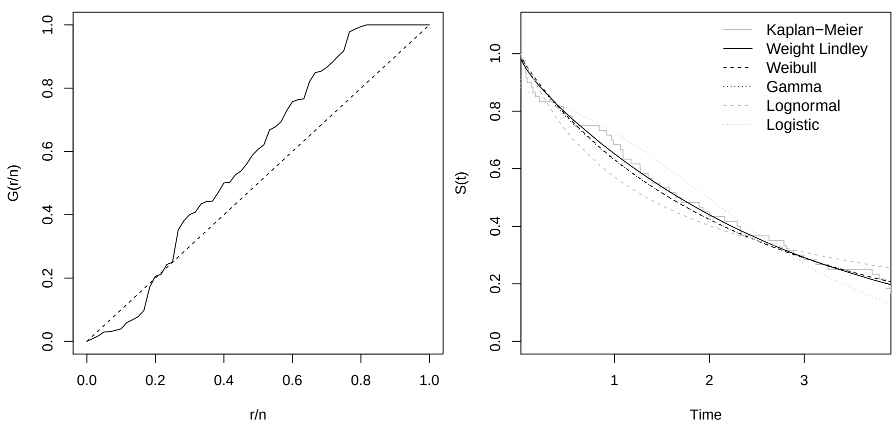

Figure 3 - TTT-plot, survival function and the hazard function adjusted by weighted Lindley distribution considering the electrical data.

Similar to first data set, comparing the empirical survival function with the adjusted by the parametric models and through the different discrimination criterion methods we observed a goodness of the fit for the WL distribution. Moreover, basead on the TTT-plot there is an indication that the hazard function has bathtub shape. This result is confirmed by the MLEs since $\hat{\phi} = 0.6764$ (bathtub shape when $0 < \phi < 1$). Therefore, considering our proposed methodology the data related to the electrical devices can be described by the WL distribution.

6 Final Comments

In this paper we presented the maximum likelihood estimators for the parameters of the weighted Lindley distribution considering the most common types of censoring, such as the type I, type II and random censoring mechanism.

An extensive numerical simulation study was conducted in order verify the performance of our proposed methodology in which is also fully illustrated with two real data set. We proved that using our method good estimates of the parameters of WL distribution were obtained. These results are of great practical interest since this will enable the use of the weighted Lindley distribution in various application issues.

There are a large number of possible extensions of the current work. The presence of covariates, as well as of long-term survivals, is very common in practice. Our approach should be investigate in both contexts. A possible approach is to consider the regression schemes adopted by Achcar and Louzada-Neto (1992) and Perdona and Louzada-Neto (2011), respectively.

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- RESUMO:
Neste trabalho apresentamos os estimadores de máxima verossimilhança para os parâmetros da distribuição weighted Lindley considerando diferentes tipos de censura, tais como, censura tipo I, tipo II e censura aleatória. Estudos de simulação numérica também foram apresentados buscando-se verificar a eficiência da metodologia proposta. Por fim, ilustramos nossa metodologia em dois conjuntos de dados reais.

- PALAVRAS-CHAVE: Distribuição weighted de Lindley; estimadores de máxima verossimilhança; dados censurados; censura aleatória.

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